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Optimal Portfolio Selection for a Defined Contribution Pension Fund with Return Clauses of Premium with Predetermined Interest Rate under Mean-variance Utility

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ABSTRACT

This paper investigates the optimal investment strategies for a defined contribution pension fund with return clauses of premiums with interest under the mean-variance criterion. Using the actuarial symbol, we formalize the problem as a continuous time mean-variance stochastic optimal control. The pension fund manager considers investments in risk and risk-free assets to increase the remaining accumulated funds to meet the retirement needs of the remaining members. Using the variational inequalities methods, we established an optimized problem from the extended Hamilton–Jacobi–Bellman Equations and solved the optimized problem to obtain the optimal investment strategies for both risk-free and risky assets and also the efficient frontier of the pension member. Furthermore, we evaluated analytically and numerically the effect of various parameters of the optimal investment strategies on it. We observed that the optimal investment strategy for the risky asset decreases with an increase in the risk-averse level, initial wealth, and the predetermined interest rate.

Key words: Defined contribution pension fund, HJB, optimal investment strategies, variational inequalities methods, return of premiums clauses, interest rate

INTRODUCTION

The investment strategy is one of the ways in which financial institutions select how the investment is made to achieve maximum profit and reduce the risk involved in such an investment. There are several ways in which financial institutions invest their funds; this includes risk-free asset (cash), coupon bond, and risky asset (stock). Portfolio optimization and risk management as an area of study have received a fast-growing attention, and it is important in the field of mathematical finance. The optimal investment strategy is a way of investing in different assets for optimal profit. This is applied in most financial institutions such as pension boards, insurance companies, and banks defined contribution (DC) scheme is a scheme that requires members to pay a certain proportion of their income to the pension managers who will help plan for their retirement. It is important in the retirement income system in most countries and is growing every day to get most workers involved in it. The DC scheme as compared to the defined benefit scheme is somehow new but forms a determining factor of the old age income adequacy for future retirees. This system underscores the need to understand better the risks that affect the income provided by this plan.

Much has been studied about optimal investment strategies in DC pension fund some of which include[1] who studied an asset allocation problem under a stochastic interest rate. Boulier *et al.*[2] studied optimal investment for DC with stochastic interest rate and Battocchio and Menoncin[3] considered a case where the interest rate was Vasicek model, Gao,[1] Deelstra *et al.*,[4] and Chubing and Ximing[5] studied the affine interest rate which includes the Cox–Ingersoll–ross model and Vasicek model. Recently, more attention has been given to constant elasticity of variance (CEV) model in DC pension fund investment strategies.[6]

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Investigated the CEV model and the Legendre transform-dual solution for annuity contracts. Since geometric Brownian motion can be considered as a special case of the CEV model, such work extended the research of Xiao *et al*.,[6] where they applied CEV model to derive dual solution of a constant relative risk aversion (CRRA) utility function through Legendre transform, also Xiao *et al*.[6] and Gao[7] extended by obtaining solutions for investor with CRRA and constant absolute risk aversion (CARA) utility function. Blake *et al*.[8] investigated an asset allocation problem under a loss-averse preference. Cairns *et al*.[9] considered a stochastic salary income of a pension beneficiary and find the investment strategy which maximizes the expected power utility of the ratio of the terminal fund and the terminal salary. Korn *et al*.[10] investigated a utility optimization problem for a DC pension plan with a stochastic salary income and a stochastic contribution process in a regime-switching economy. Gao[11] studied optimal portfolios for DC pension plans under a CEV model. Dawei and Jingyi[12] extended the work in Gao[11] by modeling pension fund with multiple contributors where benefit payment are made after retirement, he went on to find the explicit solution for CRRA and CARA using power transformation method. Osu *et al*.[13] studied optimal investment strategies in DC pension fund with multiple contributions using Legendre transformation method to obtain the explicit solution for CRRA and CARA. Akpanibah and Samaila[14] studied the stochastic strategies of optimal investment for DC pension fund with multiple contrisbutors where they considered the rate of contribution to be stochastic. Othusitse and Xiaoping[15] considered an inflationary market. In their work, the plan member made extra contribution to amortize the pension fund; the CRRA utility function was used to maximize the terminal wealth. The most commonly used utility functions are the CRRA[1,2,4,9] and CARA.[3,7]

Recently, He and Liang[16] studied optimal investment strategy for a DC pension plan with the return of premiums clauses in a mean-variance framework,[17] studied optimal time-consistent investment strategy for a DC pension with the return of premiums clauses and annuity contracts. Li *et al*.[18] studied equilibrium investment strategy for DC pension plan with default risk and return of premiums clauses under CEV model.

This research investigates the optimal investment strategy for a DC pension fund with a return of premium clauses with interest under mean-variance criterion. We assume that the return clause includes the risk-free interest due to the death member unlike what we have in the literature where death members are only entitled to their accumulation; also we assume the remaining accumulations after return are not shared equally by the members that are alive. We use the actuarial symbol and formalize the problem as a continuous time mean-variance stochastic optimal control. We consider investment in both risk-free and risky asset to increase the remaining accumulated fund to meet up the retirement need of the remaining members. Using the variational inequalities methods, we established an optimized problem from the extended Hamilton–Jacobi–Bellman Equations and solved the optimized problem to obtain the optimal investment strategies for both risk-free and risky assets and also the efficient frontier of the pension member. We also evaluated analytically the effect of various parameters of the optimal investment strategies on it.

PRELIMINARIES

Starting with a complete and frictionless financial market which is continuously open over a fixed time interval $0 \leq t \leq T$, where T is the retirement time of a given shareholder.

Let the market be a risk-free asset (cash) and a risky asset (stock). Suppose (Ω, F, P) is a complete probability space such that Ω is a real space and P a probability measure $\{W_0(t): \geq 0\}$ is a standard Brownian motion. F is the filtration and denotes the information generated by the Brownian motion $\{W_0(t)\}$.

Let $B_t(t)$ denote the price of the risk-free asset, and its model is given as:

$$\frac{dB_t(t)}{B_t(t)} = r\,dt, \tag{1}$$

Where r is the risk-free interest rate, which is predetermined.

Let $S_t(t)$ denote the price of the risky asset and its dynamics is given based on its stochastic nature and the price process is described as follows:

$$\frac{dS_t(t)}{S_t(t)} = \vartheta dt + \sigma dW_0 \tag{2}$$

Where ϑ an expected instantaneous rate of return of the risky asset and satisfies the general condition. $\vartheta > r$. σ is the instantaneous volatility of the risky asset.

Let q be the premium received at a given time, which is known, ω_0 represent the initial age of accumulation phase, T is the time frame of the accumulation phase such that ω_0+T is the end age. The actuarial symbol $\delta_{\left(\frac{1}{n}\right),\omega_0+t}$ is the mortality rate from time t to $t+\frac{1}{n}$, tq is the premium accumulated at time t, $tqd_{\left(\frac{1}{n}\right),\omega_0+t}$ is the premium returned to the death members. Furthermore, we assume that after return of premium to death members, the remaining accumulations are not shared equally unlike in.[16] Second, we assume that apart from the accumulated fund of the death member, a certain interest is paid as well from the investment in the risk-free asset since the interest rate is predetermined.

Let ρ represents the proportion of the wealth to be invested in risky assets and 1-ρ, the proportion to be invested in the risk-free asset.

Considering the time interval $[t,t+\frac{1}{n}]$, the differential form associated with the fund size is given as:

$$X\left(t+\frac{1}{n}\right)=X(t)\left(\rho\frac{S_{t+\frac{1}{n}}}{S_t}+(1-\rho)\frac{B_{t+\frac{1}{n}}}{B_t}\right)+ q\left(\frac{1}{n}\right)-tq\delta_{\left(\frac{1}{n}\right),\omega_0+t}-X(t)(1-\rho)\frac{dB_t(t)}{B_t(t)}\delta_{\left(\frac{1}{n}\right),\omega_0+t} \quad (3)$$

$$X\left(t+\frac{1}{n}\right)=X(t)\begin{pmatrix}\rho\left(\frac{S_{t+\frac{1}{n}}}{S_t}-\frac{S_t}{S_t}+\frac{S_t}{S_t}\right)+(1-\rho)\\ \left(\frac{B_{t+\frac{1}{n}}}{B_t}-\frac{B_t}{B_t}+\frac{B_t}{B_t}\right)\end{pmatrix}+ q\left(\frac{1}{n}\right)-tq\delta_{\left(\frac{1}{n}\right),\omega_0+t}-X(t)(1-\rho)\frac{dB_t(t)}{B_t(t)}\delta_{\left(\frac{1}{n}\right),\omega_0+t} \quad (4)$$

$$X\left(t+\frac{1}{n}\right)=X(t)\begin{pmatrix}\rho\left(\frac{S_{t+\frac{1}{n}}}{S_t}-\frac{S_t}{S_t}+\frac{S_t}{S_t}\right)+(1-\rho)\\ \left(\frac{B_{t+\frac{1}{n}}}{B_t}-\frac{B_t}{B_t}+\frac{B_t}{B_t}\right)\end{pmatrix}+ q\left(\frac{1}{n}\right)-tq\delta_{\left(\frac{1}{n}\right),\omega_0+t}-X(t)(1-\rho)\frac{dB_t(t)}{B_t(t)}\delta_{\left(\frac{1}{n}\right),\omega_0+t} \quad (5)$$

$$X\left(t+\frac{1}{n}\right)-X(t)=X(t)\begin{pmatrix}\rho\left(\frac{S_{t+\frac{1}{n}-S_t}}{S_t}\right)+(1-\rho)\\ \left(\frac{B_{t+\frac{1}{n}-B_t}}{B_t}\right)\end{pmatrix}+ q\left(\frac{1}{n}\right)-tq\delta_{\left(\frac{1}{n}\right),\omega_0+t}-X(t)(1-\rho)\frac{dB_t(t)}{B_t(t)}\delta_{\left(\frac{1}{n}\right),\omega_0+t} \quad (6)$$

$$\delta_{\left(\frac{1}{n}\right),\omega_0+t}=1-\exp\{-\int_0^{\frac{1}{n}}\mu(\omega_0+t+s)ds\}= \mu(\omega_0+t)\frac{1}{n}+O(\frac{1}{n}) \quad (7)$$

$$n\to\infty,\delta_{\left(\frac{1}{n}\right),\omega_0+t}=\mu(\omega_0+t)dt,\ q\left(\frac{1}{n}\right)\to qdt,\frac{S_{t+\frac{1}{n}-S_t}}{S_t}\to\frac{dS_t(t)}{S_t(t)},\frac{B_{t+\frac{1}{n}-B_t}}{B_t}\to\frac{dB_t(t)}{B_t(t)} \quad (8)$$

Substituting (8) into (6) we have

$$dX(t)=X(t)\left(\rho\left(\frac{dS_t(t)}{S_t(t)}\right)+(1-\rho)\left(\frac{dB_t(t)}{B_t(t)}\right)\right)+ qdt-tq\mu(\omega_0+t)dtX(t) \quad (9)$$
$$(1-\rho)\frac{dB_t(t)}{B_t(t)}\mu(\omega_0+t)dt$$

$dX(t) = X(t)(\rho(\vartheta dt + \sigma dW_0) + (1-\rho)(rdt)) + qdt-tq\mu(\omega_0+t)dt-X(t)(1-\rho)rdt(\omega_0+t)\,dt$ (10)

$$dX(t)=\left\{\begin{matrix}X(t)\left(\begin{matrix}\rho\left[\vartheta-r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right]+\\ r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\end{matrix}\right)+\\ q\left(\frac{\omega-\omega_0-2t}{\omega-\omega_0-t}\right)\end{matrix}\right\}dt$$
$$+\rho X(t)\sigma dW_0\quad X(0)=x_0 \quad (11)$$

Where

$$\mu(t)=\frac{1}{\omega\text{-}t}0\le t<\omega \quad (12)$$

μ(t)is the force function and is the maximal age of the life table.

METHODOLOGY

Considering the pension wealth and the volatility of the accumulation, the surviving members will want to maximize the fund size and at the same time minimize the volatility of the accumulated wealth. Hence, we formulate the optimal investment problem under the mean-variance criterion as follows:

$$\sup_{\rho}\left\{E_{t,x}X^{\rho}(T)-Var_{t,x}X^{\rho}(T)\right\} \tag{13}$$

Our interest, here, is to obtain the optimal investment strategies for both the risk-free and risky asset using the mean-variance utility function.

Applying variational inequality method in He and Liang[16] and Björk and Murgoci,[19] the mean-variance control problem in equation (13) is equivalent to the following Markovian time inconsistent stochastic optimal control problem with value function M (t,x).

$$\begin{cases} N(t,x,\rho)=E_{t,x}[X^{\rho}(T)]-\dfrac{\gamma}{2}Var_{t,x}[X^{\rho}(T)] \\ N(t,x,\rho)=E_{t,x}[X^{\rho}(T)]-\dfrac{\gamma}{2}\left(E_{t,x}[X^{\rho}(T)^2\right]-(E_{t,x}[X^{\rho}(T)])^2) \\ M(t,x)=\sup_{\rho}N(t,x,\rho) \end{cases} \tag{14}$$

Following[16] the optimal investment strategy ρ* satisfies:

$M(t,x)=\sup_{\rho}N(t,x,\rho^*)$ (15)

γ is a constant representing risk aversion coefficient of the members

Let $u^{\rho}(t,x)=E_{t,x}[X^{\rho}(T)]$, $v^{\rho}(t,x)=E_{t,x}[X^{\rho}(T)^2]$ then $M(t,x)=\sup_{\rho}f(t,x,u^{\rho}(t,x),v^{\rho}(t,x))$

Where,

$$f(t,x,u,v)=u-\frac{\gamma}{2}(v-u^2) \tag{16}$$

Theorem 3.1 (verification theorem). If there exist three real functions F, G, H: [0,T]×R→R satisfying the following extended Hamilton–Jacobi–Bellman Equation:

$$\begin{cases} \sup_{\rho}\left\{F_t-f_t+(F_x-f_x)\left[x\left(\rho\left[\vartheta-r\left(\dfrac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right]+r\left(\dfrac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)+q\left(\dfrac{\omega-\omega_0-2t}{\omega-\omega_0-t}\right)\right]+\dfrac{1}{2}(F_{xx}-U_{xx})\rho^2x^2\sigma^2\right\} \\ F(T,x)=f(t,x,x,x^2) \end{cases} \tag{17}$$

Where,

$U_{xx}=f_{xx}+2f_{xu}u_x+2f_{xv}v_x+f_{uu}u_x^2+2f_{uv}u_xv_x+f_{vv}v_x^2=\gamma u_x^2$

$$\begin{cases} \left\{G_t+G_x\left[x\left(\rho\left[\vartheta-r\left(\dfrac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right]+r\left(\dfrac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)+q\left(\dfrac{\omega-\omega_0-2t}{\omega-\omega_0-t}\right)\right]+\dfrac{1}{2}G_{xx}\rho^2x^2\sigma^2\right\}=0 \\ G(T,x)=x \end{cases} \tag{18}$$

$$\begin{cases} \left\{H_t+H_x\left[x\left(\rho\left[\vartheta-r\left(\dfrac{\omega-\omega_0-t-1}{ù-ù_0-t}\right)\right]+r\left(\dfrac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)+q\left(\dfrac{\omega-\omega_0-2t}{\omega-\omega_0-t}\right)\right]+\dfrac{1}{2}H_{xx}\rho^2x^2\sigma^2\right\}=0 \\ H(T,x)=x^2 \end{cases} \tag{19}$$

Then, $M(t,x)=F(t,x)$, $u^{\rho*}=G(t,x)$, $v^{\rho*}=H(t,x)$ for the optimal investment strategy ρ*

Proof:

The details of the proof can be found in He and Liang[20-23]

Our focus now is to obtain the optimal investment strategies for both risky and riskless asset as well as the efficient frontier by solving.[5,18,19]

Recall that $f(t,x,u,v)=u-\dfrac{\gamma}{2}(v-u^2)$

$$\begin{aligned} &f_t=f_x=f_{xx}=f_{xu}=f_{xv}=f_{uv}= \\ &f_{vv}=0, f_u=1+\gamma u, f_{uu}=\gamma, f_v=-\frac{\gamma}{2} \end{aligned} \tag{20}$$

Substituting (20) into (17) and differentiating (17) with respect to ρ and solving for ρ we have:

$$\rho^* = -F_x\left[\frac{\left(\vartheta - r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\left(F_{xx}-\gamma G_x^2\right)x\sigma^2}\right] \quad (21)$$

Substituting (21) into (17) and (18) we have

$$F_t+F_x\left[rx\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)+q\left(\frac{\omega-\omega_0-2t}{\omega-\omega_0-t}\right)\right]- F_x^2\frac{\left[\vartheta-r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right]^2}{\left(F_{xx}-\gamma G_x^2\right)\sigma^2}=0 \quad (22)$$

$$G_t+G_x\left[rx\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)+q\left(\frac{\omega-\omega_0-2t}{\omega-\omega_0-t}\right)\right] F_x\frac{\left[\vartheta-r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right]^2}{\left(F_{xx}-\gamma G_x^2\right)\sigma^2}+ G_{xx}F_x^2\frac{\left[\vartheta-r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right]^2}{2\left(F_{xx}-\gamma G_x^2\right)\sigma^2}=0 \quad (23)$$

Next, we assume a solution for F(t,x) and G(t,x) as follows:

$$\begin{cases} F(t,x)=A_1(t)x+A_2(t)A_1(T)=1,\ A_2(T)=0\\ G(t,x)=B_1(t)x+B_2(t)\ B_1(T)=1,\ B_2(T)=0\\ F_t=xA_{1t}(t)+A_{2t}(t),\ F_x=A_1(t),\ F_{xx}=0,\\ G_t=xB_{1t}(t)+B_{2t}(t),\ G_x=B_1(t),\ G_{xx}=0 \end{cases} \quad (24)$$

Substituting (24) into (22) and (23)

$$\begin{cases} A_{1t}(t)+r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)A_1(t)=0\\ A_{2t}(t)+A_1(t)q\left(\frac{\omega-\omega_0-2t}{\omega-\omega_0-t}\right)+\\ A_1^2(t)\frac{\left[\vartheta-r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right]^2}{2\gamma B_1^2(t)\sigma^2}=0 \end{cases} \quad (25)$$

$$\begin{cases} B_{1t}(t)+r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)B_1(t)=0\\ B_{2t}(t)+B_1(t)q\left(\frac{\omega-\omega_0-2t}{\omega-\omega_0-t}\right)+\\ A_1(t)\frac{\left[\vartheta-r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right]^2}{\gamma A_1(t)\sigma^2}=0 \end{cases} \quad (26)$$

Solving (25) and (26), we have

$$A_1(t)=\left(\frac{\omega-\omega_0-T}{\omega-\omega_0-t}\right)^r e^{r(T-t)} \quad (27)$$

$$B_1(t)=\left(\frac{\omega-\omega_0-T}{\omega-\omega_0-t}\right)^r e^{r(T-t)} \quad (28)$$

$$A_2(t)=\frac{1}{2\gamma\sigma^2}\left\{\begin{matrix}(\vartheta-r)^2(T-t)-(\vartheta-r)\\ \ln\left(\frac{\omega-\omega_0-T}{\omega-\omega_0-t}\right)^{2r}+\frac{r^2}{\omega-\omega_0-t}\end{matrix}\right\}+ qe^{rT}(\omega-\omega_0-T)^r\int_t^T\frac{\omega-\omega_0-2\tau}{(\omega-\omega_0-\tau)^{1+r}}e^{-r\tau}d\tau \quad (29)$$

$$B_2(t)=\frac{1}{\gamma\sigma^2}\left\{\begin{matrix}(\vartheta-r)^2(T-t)-(\vartheta-r)\ln\\ \left(\left(\frac{\omega-\omega_0-T}{\omega-\omega_0-t}\right)^{2r}+\frac{r^2}{\omega-\omega_0-t}\right)\end{matrix}\right\}+ qe^{rT}(\omega-\omega_0-T)^r\int_t^T\frac{\omega-\omega_0-2\tau}{(\omega-\omega_0-t)^{1+r}}e^{-r\tau}d\tau \quad (30)$$

$$F(t,x)=x\left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^r e^{-r(T-t)}+ \frac{1}{2\gamma\sigma^2}\left\{\begin{matrix}(\vartheta-r)^2(T-t)-(\vartheta-r)\ln\\ \left(\left(\frac{\omega-\omega_0-T}{\omega-\omega_0-t}\right)^{2r}+\frac{r^2}{\omega-\omega_0-t}\right)\end{matrix}\right\}+ qe^{rT}(\omega-\omega_0-T)^r\int_t^T\frac{\omega-\omega_0-2\tau}{(\omega-\omega_0-\tau)^{1+r}}e^{-r\tau}d\tau \quad (31)$$

$$G(t,x) = x\left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^{r} e^{-r(T-t)} + \frac{1}{\gamma\sigma^2}\left\{\begin{array}{l}(\vartheta-r)^2(T-t)-(\vartheta-r)\ln\\ \left(\left(\frac{\omega-\omega_0-T}{\omega-\omega_0-t}\right)^{2r}+\frac{r^2}{\omega-\omega_0-t}\right)\end{array}\right\} + qe^{rT}(\omega-\omega_0-T)^{r}\int_t^T \frac{\omega-\omega_0-2\tau}{(\omega-\omega_0-\tau)^{1+r}}e^{-r\tau}d\tau \tag{32}$$

Hence, (21) becomes

$$\rho^{*} = \left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^{r}\left[\frac{\left(\vartheta-r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\gamma x\sigma^2}\right]e^{-r(T-t)} \tag{33}$$

Next, we compute the efficient frontier:

$$Var_{t,x}[X^{\rho^*}(T)] = E_{t,x}[X^{\rho}(T)^2] - (E_{t,x}[X^{\rho}(T)])^2$$

$$Var_{t,x}[X^{\rho^*}(T)] = \frac{2}{\gamma}(G(t,x)-F(t,x))$$

$$Var_{t,x}[X^{\rho^*}(T)] = \frac{1}{\gamma^2\sigma^2}\left\{\begin{array}{l}(\vartheta-r)^2(T-t)-(\vartheta-r)\\ \ln\left(\frac{\omega-\omega_0-T}{\omega-\omega_0-t}\right)^{2r}+\frac{r^2}{\omega-\omega_0-t}\end{array}\right\} \tag{34}$$

$$\frac{1}{\gamma} = \sigma\sqrt{\frac{Var_{t,x}[X^{\rho^*}(T)]}{\left\{\begin{array}{l}(\vartheta-r)^2(T-t)-(\vartheta-r)\ln\\ \left(\left(\frac{\omega-\omega_0-T}{\omega-\omega_0-t}\right)^{2r}+\frac{r^2}{\omega-\omega_0-t}\right)\end{array}\right\}}} \tag{35}$$

$$E_{t,x}[X^{\rho}(T)] = G(t,x)E_{t,x}[X^{\rho^*}(T)] = x\left(\frac{\omega-\omega_0-T}{\omega-\omega_0-t}\right)^{r}e^{r(T-t)} + \frac{1}{\gamma\sigma^2}\left\{\begin{array}{l}(\vartheta-r)^2(T-t)-(\vartheta-r)\ln\\ \left(\left(\frac{\omega-\omega_0-T}{\omega-\omega_0-t}\right)^{2r}+\frac{r^2}{\omega-\omega_0-t}\right)\end{array}\right\} + qe^{rT}(\omega-\omega_0-T)^{r}\int_t^T\frac{\omega-\omega_0-2\tau}{(\omega-\omega_0-\tau)^{1+r}}e^{-r\tau}d\tau \tag{36}$$

Substitute (35) in (36), we have:

$$E_{t,x}[X^{\rho^*}(T)] = x\left(\frac{\omega-\omega_0-T}{\omega-\omega_0-t}\right)^{r}e^{r(T-t)} + qe^{rT}(\omega-\omega_0-T)^{r}\int_t^T\frac{\omega-\omega_0-2\tau}{(\omega-\omega_0-\tau)^{1+r}}e^{-r\tau}d\tau + \frac{1}{\sigma}\sqrt{\left(\left\{\begin{array}{l}(\vartheta-r)^2(T-t)-(\vartheta-r)\ln\\ \left(\left(\frac{\omega-\omega_0-T}{\omega-\omega_0-t}\right)^{2r}+\right.\\ \left.\frac{r^2}{\omega-\omega_0-t}\right)\end{array}\right\}Var_{t,x}[X^{\rho^*}(T)]\right)} \tag{37}$$

RESULT ANALYSIS

Next, we study the effect of the various parameters on the optimal investment strategy

Proposition 1

Assume $\left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^{r} > 0, e^{-r(T-t)} > 0$ and

$$\left[\frac{\left(\vartheta-r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\gamma x\sigma^2}\right] > 0$$

Then $\frac{d\rho^*}{d\gamma} < 0$

Proof:

$$\frac{d\rho^*}{d\gamma} = -\left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^{r}\left[\frac{\left(\vartheta-r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\gamma^2 x\sigma^2}\right]e^{-r(T-t)}$$

But $\left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^{r} > 0, e^{-r(T-t)} > 0$, and

$$\left[\frac{\left(\vartheta - r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\gamma^2 x\sigma^2}\right] > 0$$

Therefore $\frac{d\rho^*}{d\gamma} < 0$

Proposition 2

Assume

$$\rho^* = \left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^r \left[\frac{\left(\vartheta - r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\gamma x\sigma^2}\right] e^{-r(T-t)}$$

such that $\left(\frac{\omega-\omega_0-T}{\omega-\omega_0-t}\right)^{-r} > 0, e^{-r(T-t)} > 0$, and

$$\left[\frac{\left(\vartheta - r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\gamma x\sigma^2}\right] > 0$$

Then, $\frac{d\rho^*}{d\gamma} < 0$

Proof:

$$\rho^* = \left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^r \left[\frac{\left(\vartheta - r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\gamma x\sigma^2}\right] e^{-r(T\ \ t)}$$

$$\frac{d\rho^*}{dr} = \frac{-\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)}{\gamma x\sigma^2}\left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^r e^{-r(T-t)}$$

$$-(T-t)\left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^r \left[\frac{\left(\vartheta - r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\gamma x\sigma^2}\right] e^{-r(T-t)} -$$

$$\left(\frac{\omega-\omega_0-T}{\omega-\omega_0-t}\right)^{-r} \ln\left(\frac{\omega-\omega_0-T}{\omega-\omega_0-t}\right)\left[\frac{\left(\vartheta - r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\gamma x\sigma^2}\right] e^{-r(T-t)}$$

$$\frac{d\rho^*}{dr} = -\left\{\begin{array}{c} \frac{\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)}{\gamma x\sigma^2}\left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^r e^{-r(T-t)} + \\ (T-t)\left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^r \left[\frac{\left(\vartheta - r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\gamma x\sigma^2}\right] e^{-r(T-t)} + \\ \left(\frac{\omega-\omega_0-T}{\omega-\omega_0-t}\right)^{-r} \ln\left(\frac{\omega-\omega_0-T}{\omega-\omega_0-t}\right) \\ \left[\frac{\left(\vartheta - r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\gamma x\sigma^2}\right] e^{-r(T-t)} \end{array}\right\}$$

Since $\left(\frac{\omega-\omega_0-T}{\omega-\omega_0-t}\right)^{-r} > 0, e^{-r(T-t)} > 0,$ and

$$\left[\frac{\left(\vartheta - r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\gamma x\sigma^2}\right] > 0$$

Hence,

$$\frac{d\rho^*}{d\gamma} < 0$$

Proposition 3

Assume $\left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^r > 0, e^{-r(T-t)} > 0$, and

$$\left[\frac{\left(\vartheta - r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\gamma x\sigma^2}\right] > 0$$

Then, $\frac{d\rho^*}{dx} < 0$

Proof:

$$\frac{d\rho^*}{dx} = -\left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^r \left[\frac{\left(\vartheta - r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{x^2\gamma\sigma^2}\right] e^{-r(T-t)}$$

However, $\left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^r > 0, e^{-r(T-t)} > 0$, and

$$\left[\frac{\left(\vartheta - r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\gamma^2 x\sigma^2}\right] > 0$$

Therefore, $\frac{d\rho^*}{dx} < 0$

Proposition 4

Assume

$$\rho^* = \left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^r \left[\frac{\left(\vartheta - r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\gamma x\sigma^2}\right] e^{-r(T-t)}$$

such that $0 < r < 1, \left(\frac{\omega-\omega_0-T}{\omega-\omega_0-t}\right)^{-r} > 0, e^{-r(T-t)} > 0$,

and $\left[\frac{\left(\vartheta - r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\gamma x\sigma^2}\right] > 0$

Then, $\frac{d\rho^*}{dt} < 0$

Proof:

$$\rho^* = \left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^r \left[\frac{\left(\vartheta - r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\gamma x\sigma^2}\right] e^{-r(T-t)}$$

$$\frac{d\rho^*}{dt} = -r\left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^{r-1} \left[\frac{\left(\vartheta - r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\gamma x\sigma^2}\right] e^{-r(T-t)} +$$

$$r\left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^{r} \left[\frac{\left(\vartheta - r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\gamma x\sigma^2}\right] e^{-r(T-t)}$$

$$+\left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^{r} \left[\frac{\left(\frac{r}{(\omega-\omega_0-t)^2}\right)}{\gamma x\sigma^2}\right] e^{-r(T-t)}$$

However,

$$r\left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^{r} \left[\frac{\left(\vartheta - r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\gamma x\sigma^2}\right] e^{-r(T-t)}$$

$$-r\left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^{r-1} \left[\frac{\left(\vartheta - r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\gamma x\sigma^2}\right] e^{-r(T-t)}$$

$$= r\left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^{r} \left[\frac{\left(\vartheta - r\left(\frac{\omega-\omega_0-t-1}{\omega-\omega_0-t}\right)\right)}{\gamma x\sigma^2}\right] e^{-r(T-t)}$$

$$\left(\left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^{r} - \left(\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\right)^{r-1}\right) > 0$$

Therefore, $\frac{d\rho^*}{dt} < 0$

NUMERICAL SIMULATIONS

In this section, we presented numerical simulations of the optimal investment strategy with respect to time and observed the effect of the various parameters of the optimal investment strategy on it using math lab programming language.

DISCUSSION

From proposition 1 and Figure 1, the optimal investment strategy increases with a decrease in the risk-aversion coefficient. The implication is that members with high risk averse will prefer to invest more in riskless asset and will reduce that of the risky asset.

Proposition 2 and Figure 2 show that the optimal investment strategy increases with a decrease in the interest rate of the risk-free asset. This implies that if the interest rate of the risk-free asset is high, the members will increase the proportion of its wealth to be invested in risk-free asset thereby reducing the proportion invested in risky asset and vice versa.

Proposition 3 shows that the optimal investment strategy decreases with an increase in the initial wealth. The implication here is that if the initial

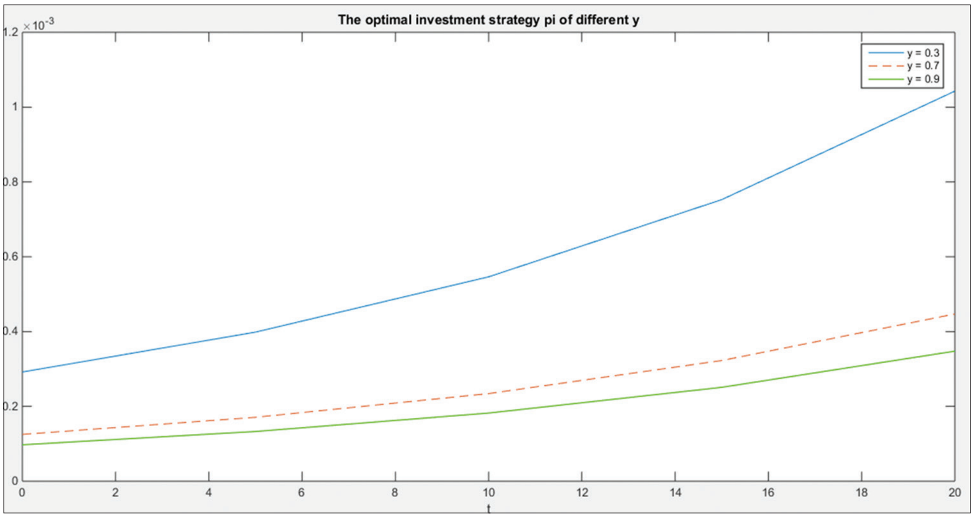

Figure 1: The optimal investment strategy with different risk averse level

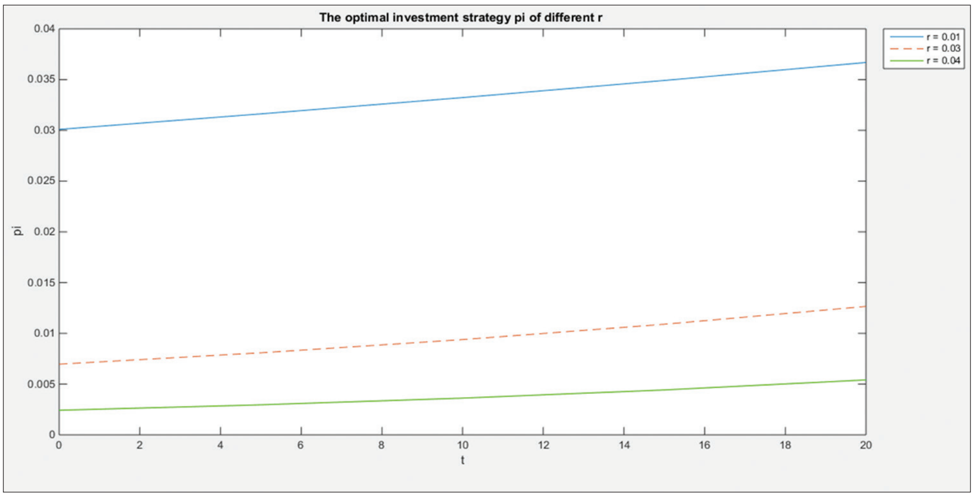

Figure 2: The optimal investment strategy with different predetermined interest rates

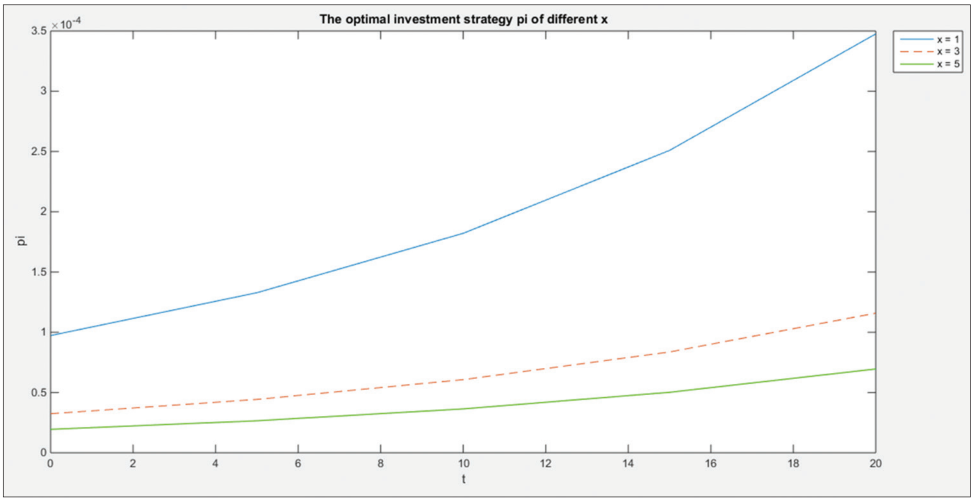

Figure 3: The optimal investment strategy with different initial wealth

wealth of the plan member is high, the member will prefer to invest more in risk free asset to minimize risk instead of investing more in risky asset but if the initial wealth is low, the member prefers taking the risk to grow the wealth by investing in risky asset.

Proposition 4 shows that the optimal investment strategy increases as time increases, the implication is that as retirement age approaches, the member is eager to grow his or her wealth hence an increase in investment in the risky asset. This is confirmed numerically from the simulation shown in Figures 1-3 that the graphs give positive slopes, indicating that as t increases, the optimal investment strategy increases also. Observed that proposition 1, 2, and 3 are confirm numerically in these figures. In general, we observe that at the beginning of the accumulated phase, the pension manager will invest more in risk-free asset because there is no return initially, but once there is a return to the death members, the fund manager will increase its investment in the risky asset to meet the retirement needs of the remaining members. Furthermore, we observe that if the return clause is with predetermined interest, the fund size will reduce even more with time. Hence, cause an increase in the optimal investment strategy.

CONCLUSION

We investigated optimal investment strategies for DC pension fund with a return of premium clauses with interest under mean-variance criterion. Using the actuarial symbol, we formalize the problem as a continuous time mean-variance stochastic optimal control. The pension fund manager considers investment in both risk-free and risky asset to increase the remaining accumulated fund to meet up the retirement need of the remaining members. Using the variational inequalities methods, we established an optimized problem from the extended Hamilton–Jacobi–Bellman Equations and solved the optimized problem to obtain the optimal investment strategies for both risk-free and risky assets and the efficient frontier of the member. We also evaluated analytically the effect of various parameters of the optimal investment strategies on it. In general, we observe that at the beginning of the accumulated phase, the pension manager will invest more in risk-free asset because there is no return initially, but once there is a return to the death members, the fund manager will increase its investment in the risky asset to meet the retirement needs of the remaining members. Furthermore, we observe that if the return clause is with risk-free interest, the fund size will reduce even more with time due to the clause, hence leading to an increase in the optimal investment strategy.

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