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On the Fixed Point Extension Results in the Differential Systems of Ordinary Differential Equations

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ABSTRACT

In this work, a fixed point $x(t)$, $t \in [a,b]$, $a \leq b \leq +\infty$ of differential system is said to be extendable to $t = b$ if there exists another fixed point $\bar{x}(t), t \in [a,c], c \geq b$ of the system (1.1) below and $\bar{x}(t) = x(t), t \in [a,b)$ so that given the system

$$x' = f(t,x);\ f\colon J \times M \to R^n$$

We aim at using the established Peano's theorem on existence of the fixed point plus Picard–Lindelof theorem on uniqueness of same fixed point to extend the ordinary differential equations whose local existence is ensured by the above in a domain of open connected set producing the result that if D is a domain of $R \times R_n$ so that $F\colon D \to R_n$ is continuous and suppose that (t_0, x_0) is a point D where if the system has a fixed point $x(t)$ defined on a finite interval (a,b) with $t \in (a,b)$ and $x(t_0) = x_0$, then whenever f is bounded and D, the limits

$$x(a^+) = \lim_{t \to a^+} x(t)$$

$$x(b^-) = \lim_{t \to b} x(t)$$

exist as finite vectors and if the point $(a, x(a^+)), (b, x(b^-))$ is in D, then the fixed point $x(t)$ is extendable to the point $t = a(t = b)$. Stronger results establishing this fact are in the last section of this work.

Key words: Fixed point, continuous differential systems, existence and uniqueness, extendability
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INTRODUCTION

Let $(X, \|.\|)$ be a real Banach space which is Frechet differentiable. If for any real operator F and $x \in X = R_n$ with t arbitrary, there exists a unique x such that

$$F(x(t)) = x(t)$$

Satisfying the differential system

$$x' = f(t,x)$$

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For $f\colon J \times M \to R^n$ continuous, J an interval R and M a subset of R_n then $x(t)$ is called a fixed point for (1.1) below.

Consider the differential system,

$$x' = f(t,x) \tag{1.1}$$

Where, $f\colon J \times M \to R_n$ is continuous, and J an interval R and M a subset of R_n, then presented below, are preliminary results on the fixed point of such ordinary differential system through Peano's and Picard's theorems, respectively, stated and also extensively used in the later section.

Theorem 1.1 (The Peano Theorem on Existence Of Solutions)[12]

Let (t_0,x_0) be a given point in $R \times R^n$. Let $J=\{t_0 - a, t_0\}$, $D = \{x \in R^n: |x - x_0| \leq b\}$ where a,b are two positive constants. Assume the following: $F: J \times D \rightarrow R^n$ is continuous with $|F(t,u)| \leq L$, $(t,u) \in J \times D$, where L is a positive constant. Then, there exists a fixed point $x(t)$ of (1.1) with the following property: $x(t)$ is defined and satisfied (1.1) on

$$S = \{t \in J : |t - t_0| \leq \alpha\}$$

with

$$\alpha = \min\left\{\alpha, \frac{b}{L}\right\}$$

Moreover,

$$x(t_0) = x_0$$

and

$$|x(t) - x_0| \leq b$$

for all

$$t \in (t_0 - \alpha, t_0 + \alpha)$$

For proof see (1)

Theorem 1.2 Picard's–Lindelof Theorem on Uniqueness

Consider system (1.1) under the assumptions of theorem (1.1). Let

$$F : J \times D \rightarrow R^n$$

satisfy $|F(t, x_1) - F(t, x_0)| \leq K|x_1 - x_0|$ for every (t,x_1), $(t,x_2) \in J \times D$, where K is a positive constant, then there exists a unique solution $x(t)$ satisfying the conclusion of theorem (1.1). The proof is in (1)

Theorem 1.3[11,12]

Let $F: (a \times b) \times D \rightarrow R^n$ be continuous, where $D = \{u \in R^n; |u - x_0| \leq r\}$, x_0 a fixed point of R^n and r, a positive constant.

Let $\{|F(t,u) - F(t,u_0)| \leq I(t)| u_1 - u_0 |r\}| F(t,u) \leq m(t)$ for every $t \in (a, b)$, $u, u_0, u_1 \in D$, where $I, m: [a, b] \rightarrow R_+$ are continuous and integrable in the improper sense on (a,b) choose the number $b \ni a_1 < b_1 < b$ so that

$$L = \int_{a^+}^{b_1} I(t)\,dt < 1, \int_{a^+}^{b_1} m(t)\,dt \leq r$$

Then, the integrable equation has unique fixed point $x(t)$ on the interval $[a,b]$. This fixed point satisfies $x(a^+) = x_0$ and the system (1.1) on the interval (a,b).

EXTENSION

This section presents the study of extendability or continuation of the fixed points whose local existence is ensured by theorem (1.1) and (1.2). In what follows, a domain is an open connected set and we have the following definition.[1-5]

Definition 2.1[1,6]

A fixed point $x(t)$, $t \in [a,b]$, $a < b < +\infty$ of system (1.1) is said to be extendable (continuable) to $t = b$ ifthereexistsanotherfixedpoint $\bar{x}(t), t \in [a,c], c \geq b$ of the system (1.1) such that $\bar{x}(t) = x(t), t \in [a,b)$.

A fixed point $x(t)$, $t \in [a,b]$, $(a < b < +\infty)$ of system (1.1) is said to be extendable (continuable) to $t = c$, $a < b < +\infty$ if it is extendable (continuable) to $t = b$ and whenever we assume that $x(t)$ is a fixed point on (a, d) for any $d \in [b,c]$ we can show that $x(t)$ is extendable (continuable) to $t = d$. Such a fixed point is extendable (continuable) $t = c$ for any $c > b$. Extension to the left can be defined in a similar manner.

Theorem 2.1[1,8]

Suppose that, D is a domain of $R \times R^n$ and $F: D \rightarrow R^n$ is continuous. Let (t_0, x_0) be a point in D and assume that the system (1.1) has a fixed point $x(t)$ defined on a finite interval (a,b) with $t \in (a,b)$ and $x(t_0) = x_0$. Then, if F is bounded on D, the limits

$$\left.\begin{aligned} x(a^+) &= \lim_{t \to a^+} x(t) \\ x(b^+) &= \lim_{t \to b^+} x(t) \end{aligned}\right\} \quad (2.1)$$

exist as finite vectors. If a point $(a, x(a^+)$, $(b, x(b^-))$ is in D, then $x(t)$ is extendable to the point $t = a(t = b)$.

Proof: To show that the first limit (2.1) exists, we first note that

$$x(t) = x_0 + \int_{x_0}^{x} F(s, x(s))\,ds, \ \ t \in (a,b) \quad (2.2)$$

Now, let $|F(t,x)| < L, (t,x) \in D$ where L is positive constant. Then, if $t_1, t_2 \in (c, b)$ we obtain

$$\left|x(t_1)-x(t_2)\right| \leq x_0 + \int_{t_2}^{t_1} \left|F\left(s, x(s)\right)\right| ds \leq L\left|t_1 - t_2\right|$$

Thus, $x(t_1) - x(t_2)$ converges to zero as t_1, t_2 converges to the point $t = a$ from the right. Applying the continuity condition for functions, we obtain our assertion. Similarly, the second limit follows same argument let us assume now that the point $(b, x(b^-))$ belongs to D and consider the function

$$\bar{x}(t) = \begin{cases} x(t), t \in (a,b) \\ x(b^-), t = b \end{cases}$$

This function is a fixed point of (1.1) on (a,b). In fact (2.2) implies

$$\bar{x}(t) = x_0 + \int_{t_0}^{c} \left|F\left(s, x(s)\right)\right| ds, t \in (a,b)$$

This, in turn, implies the existence of the left hand derivative $\bar{x}(b)$ of $\bar{x}(t)$ as $t = b$. Thus, we have

$$\bar{x}^{'}(b) = F\left(b, \bar{x}(b)\right)$$

Which completes the proof for $t = b$. A similar argument holds for $t = a$.

Theorem 2.2

Let F: $[a,b] \times R^n \to R^n$ be continuous and such that

$$\| F(t,x) \| \leq L$$

for all $(t,x) \in (a,b) \times R^n$ where L is a positive constant, then every fixed point $x(t)$ of (1.1) is extendable to the point $t = b$.

Proof: Let $x(t)$ be a fixed point of (1.1) passing through the point $(t_0, x_0) \in [a,b] \times R^n$. Assume that $x(t)$ is defined on the interval (t_0, c) where c is the same point with $c \leq b$. Then, as in the proof of theorem 2.1 above $x(c^-)$ exists and $x(t)$ is extendable to the point $t = c$. If $c = b$, the proof is complete.

If $c < b$, then Peano theorem applied on $(c,b) \times D$ with D a sufficiently large closed ball with center at $x(c^-)$ ensures the existence of a solution such that $\bar{x}(t), t \in (c,b)$ such that $\bar{x}(c) = x(c^-)$. Thus, the function.

$$x(t) = \begin{cases} x(t), t \in [t_0, c] \\ x(t), t \in (a,b) \end{cases}$$

is the required extension of $x(t)$.

Remark 2.1: It should be noted that the point $(a, x(a'))$ is not in D, but $F(c, x(a'))$ can be defined so that F is continuous at $(a, x(a'))$, then $x(t)$ is extendable to $(a, x(a'))$. A similar situation exists at $(b, x(b^-))$. The following extension theorem is needed for the proof of theorem 2.3

Remark 2.2: The next theorem assures the boundedness of every fixed point through a certain point in a certain sense and hence implies extendability.

Theorem 2.3

Let F: $[a,b] \times M \to R^n$ be continuous, where M is closed ball $S = \{u \in R: |u| \leq r\}$, $r > 0$ (or R'). Assume that, $(t_0, x_0) \in [a,b] \times M$ is given and that every fixed point $x(t)$ of (1.1) passing through (t_0, x_0) satisfies $|x(t)| < \lambda$ as long as it exists to the right of t_0. Here, $0 < \lambda < r$ (or $0 < \lambda < \infty$). Then, every fixed point $x(t)$ of (1.1) with $x(t_0) = x_0$ is extendable to the point $t = b$.

Proof

We give the proof for $M = S$ if $M = R$ the same proof holds and it is even easier. Let $x(t)$ be a fixed point of (1.1) with $x(t_0) = x_0$ and assume that $x(t)$ is defined on (t_0, c) with $c < b$. Since F is continuous on $[a,b] \times S_\lambda$ where

$$S = \left\{u \in R^n : u \leq \lambda\right\}$$

There exists $L > 0$ such that $\| F(t,x) \| \leq L$ for all $(t,x) \in [a,b] \times S$.

Now, consider the function

$$F(t,x) = \begin{cases} F(t,x), (t,x) \in [a,b] \times S_\lambda \\ \dfrac{\lambda}{\| x \|} F\left(t, \dfrac{\lambda x}{\| x \|}\right), \ t \in [a,b] \end{cases}$$

It is easy to see that F is continuous and such that $\| F(t,x) \| \leq L$ $[a,b] \times R^n$. Consequently, theorem 2.2 above implies that fixed point of the system

$$x^{'} = f(t,x) \qquad (2.3)$$

is extendable to $t = b$. Naturally, $x(t)$ is a fixed point defined on (t,c) because $F_{\lambda =} F$ for $\| x \| \leq \lambda$.

Therefore, there exists fixed point $x(t),\ t \in (t_0,b)$ of (2.1) such that $x(t) = \bar{x}(t), t \in [t_0, c)$. Assume that, there exists $t \in [c,b]$ such that $\| x(t) \| = \lambda$ then for some $t \in [c,t]$, $\|x(t)\| = \lambda$ for all $t \in [t_0,t_2]$. Obviously $x(t)$ satisfies that system (1.1) on $[t_0,t_2]$. This is a contradiction to our assumption. Thus, $\| x(t) \| < \lambda$ for all $t \in [t_0,b]$ which implies that $x(t)$ is extendable to the point $t = b$.

MAIN RESULTS

Here, we discuss the existence of fixed points on R_+ by the approach of extension of fixed points R to R_+. This we achieve more easily by the following results below. [7, 9, 10]

Lemma 3.1

Let $x(t), t \in [t_0, t_1], 0 \le t_0 \le t_1 \le +\infty$ be a fixed point of the system (3.1). Then, $x(t)$ is extendable to the point $t = t_1$ if and only if it is bounded on $[t_0,t_1]$.

Definition 3.1

Let $x(t),\ t \in (t_0,\mathrm{T}),\ 0 \le t_0 \le \mathrm{T} +\infty$ be a fixed point of the system (3.1). Then, $x(t)$ is said to be non-extendable (non continuable) to the right if T equals $+\infty$. or if $x(t)$ cannot be extended (continued) to the point T.

Theorem 3.1

Let $x(t),\ t \in [t_0,t_1],\ t_1 > t_0 \ge 0$ be an extendable to the right fixed point of the system (3.1) then there exists a non-extendable to the right solution of (3.1) which extends $x(t)$ that is a solution $y(t)$, $y \in (t_0,t_1)$ such that $t_0 > t_1$ $y(t) = x(t),\ t \in [t_0,t_1]$ and $y(t)$ is non-extendable to the right. Here, t_0 may equal $+\infty$.

Proof

It suffices to assume that $t_0 > 0$. Let $Q = (0,\infty) \times R^n$ and for $m = 1,2,\ldots,$ let $Q_m = \left\{(t,u) \in Q : t^2 + (u)^2 \le m, t \ge \frac{1}{m}\right\}$. Then, $Q_m \subset Q_{m+1}$ and $\cup\, Q_m = Q$.
Furthermore, each Q_m is a compact subset of Q. By lemma (3.1), a solution $y(t), t \in (t,T)$ is non-continuable or non-extendable to the right. If its graph $[t,y(t)]:\ t \in (t,T)$ intersects all the sets Q_m. We are going to construct such a fixed point $y(t)$ which extends $x(t)$. Since $x(t)$ is extendable to the right, we may consider it defined and continues in the interval $[t_0,t_1]$. Now, since the graph $Q = (t, x(t)),\ t \in [t_0,t_1]$ is compact, there exists an integer m_1 such that $G \in Q_m$. If the number $\alpha > 0$ is sufficiently small, then for every $(\alpha,u) \in Q_m$ the set

$$M_{\alpha,u} = \left\{(t,x) \in R^{n+1} : t - \alpha \le \alpha - u \le \alpha\right\}$$

is contained in the set Q_{m+1}. Let $[(t,x) \le k]$ on the set Q_{m+1} where k is a positive constant. By Peano's theorem, for every point $\{(t,u) \in Q_m\}$, there exists a fixed point $x(t)$ of the system (3.1) such that $x(t) = u$ defined on the interval $[\alpha,\alpha + \beta]$ with $\beta = [\alpha,\alpha k^{-1}]$. This number β does not depend on the particular point $(\alpha,u) \in Q$. Consequently, since point $(t,x(t)) \in Q_m$, there exists a fixed point $x(t)$ of (3.1) which continues $x(t)$ to the point $t_0 + \alpha\beta$. Repeating this process, we shall essentially have a fixed point $x(t)$ (α a positive integer) of the system (3.1) which continues $x(t)$ to the point $t_0 + \alpha\beta$ and has a graph in the set Q_{m+1} but not entirely inside the set Q_m. In this set Q_{m+1}, we repeat the extension process as in the set Q_m. Thus, we eventually obtain a fixed point $y(t)$ which intersects all the sets Q_m, $m \ge m_1$ for some m_i and in a non-continuable extension of the fixed point $x(t)$.

Theorem 3.2

Let $x(t), t \in (t_0,T), (0 \le t_0 < T < +\infty)$ be a non-extendable to the right fixed point of (1.1). Then,

$$\lim_{t \to T^-} \| x(t) \| = +\infty$$

Proof

Assume that, our assertion is false. Then, there exists an increasing sequence $\{t_m\}_{m=1}^{\infty}$ such that $t_0 \le t_m < T, \lim_{m\to\infty} t_m = T$ and $\lim_{m\to\infty} x'(t_m) < +\infty$. Since $\{x(t_m)\}, m = 1,2,\ldots,$ is bounded, there exists a subsequence $\{x'(t_m)\}$ such that $x(t_m) \to y$ as $m \to \infty$ with $\| y \| = L$ and t'_m increasing. Let M be a compact subset of $R_+ \times R^n$ such that the point (T_x,y) is an interior point of M. Then, we may show that for infinitely many m, there exists $\bar{t}_m$ such that

$$t'_m < T_m < t_{m+1},\ (t_m, x(t_m)) \in M$$

Where, M denotes the boundary of M. In fact, if this was not the case, then there would be $\varepsilon \in (0,T)$ such that $(t,x(t))$ belongs to the interior of M for all x with $T - \varepsilon < t < T$. However, then, lemma (3.1) above implies the extendability of $x(t)$ beyond T, which is a contradiction. Let theorem (3.1) above holds for a subsequence $\{m'\}$ of the positive integer that is

$$t_m' < \bar{t}_m < t_{m+1}'; (\bar{t}_{m'}, x(\bar{t}_m) \in \delta_m$$

Then, we have

$$\lim_{m\to\infty} (\bar{t}_m, x(\bar{t}_m)) = (T, y)$$

This is a consequence of the fact that $\bar{t}_m \to T$ as $m \to \infty$ and the inequality $\| x(t_m) - x(\bar{t}_m) \| \leq \mu \| \bar{t}_m - t_m \|$ where μ is a bound for F or M. However, δ_m is a closed set. Thus, the point $(T,y) \in M$ is a contradiction to our assumption and this completes the proof.

Theorem 3.3

Let $V: R_+ \times R^n \to R$ be a Lyapunov function satisfying

$$V(t,u) \leq (t,u), (t,u) \in R_+ \times R^n \tag{3.1}$$

$$V(t,u) \to +\infty$$

and $V(t,u) \to +\infty$ as $u \to R$

Uniformly with respect to t lying in any compact set. Here, $Y: R_+ \times R$ is continuous and such that for every $(t_0,u_0) \in R_+ \times R$, the problem

$$U = Y(t,u) + \varepsilon, u(t_0) = u_0 + \varepsilon \tag{3.2}$$

Has a maximal fixed point defined on $(t_0, +\infty)$. Then, every fixed point of (1.1) is extendable to $+\infty$.

Proof

Let (t_0,T) be the maximal interval of existence of fixed point of (1.1) and assume that $T < +\infty$. Let $y(t)$ be the maximal fixed point (3.1) with $y(t_0) = V(t_0,x(t_0))$. Then, since

$$Du(t) \leq Y(t, v(t)), t \in (t_0, t_1 + \alpha) \backslash$$
$$S : s\text{accountable set} \tag{3.3}$$

We have

$$V(t, x(t)) \leq Y(t), t \in [t_0, T)$$

On the other hand, since $x(t)$ is non-extendable to the right fixed point, we have

$$\lim_{t\to r} \| x(t) \| = +\infty$$

This implies that Vt converges to $+\infty$ as $t \to T$, but (3.3) implies that

$$\lim\sup V(t, x(r)) \leq Y(T)$$

As $t \to T$. Thus, $T = +\infty$

Corollary 3.1

Assume that, there exists $\alpha > 0$ such that

$$\| F(t,u) \| \leq Y \in R, \| u \| > \alpha$$

Where, $Y: R_+ \times R_+ \to R_+$ is such that for every $(t_0,u_0) \in R_+$ the problem (4) has a maximal fixed point defined on (t_0,∞). Then, every solution of (1.1) is extendable to $+\infty$

Proof

Here, it suffices to take $V(t,u) = u$ and obtain

$$V_\varepsilon(t, x(t)) = \lim_{h\to 0} \frac{x(t) + hF(t, x(t)) - x(t)}{h}$$
$$\leq \| F(t, x(t)) \| \leq Y(t, V(t, x(t)))$$

Provided that $\| x(t) \| > \alpha$. Now, let $x(t), t \in (t_0,T)$ be a fixed point non-continuable to the right fixed point such that $R < +\infty$. Then, for t sufficiently close to T from left, we have $\| x(t) \| > \alpha$. However, if $Y: R_+ \times R \to R$ be continuous and $(t_0,u_0) \in R_+ \times R_\alpha \in (0, +\infty)$ be the maximal fixed point of (4) in the interval $(t_0,t_1 + \alpha)$, let $V:(t_0,t_1 + \alpha) \to R$ be continuous and such that $u(t_0) \leq u_0$

$$Du(t) \leq Y(t, v(t)), t \in (t_0, t_1 + \alpha) \backslash S$$

And S is a countable set then

$$u(t) \leq s(t), t \in (t_0, t_1 + \alpha)$$

This implies that every fixed point $x(t)$ of (1.1) is extendable and hence the proof.

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