ISSN 2581-3463

AJMS
Asian Journal of Mathematical Sciences

RESEARCH ARTICLE

On Application of the Fixed-Point Theorem to the Solution of Ordinary Differential Equations

Eziokwu C. Emmanuel[1], Akabuike Nkiruka[2], Agwu Emeka[1]

[1]Department of Mathematics, Michael Okpara University of Agriculture, Umudike Abia State, Nigeria,
[2]Department Mathematics and Statistics, Federal Polytechnic, Oko, Anambra State, Nigeria

Received: 25-12-2018; Revised: 27-01-2019; Accepted: 20-03-2019

ABSTRACT

We know that a large number of problems in differential equations can be reduced to finding the solution *x* to an equation of the form *Tx*=*y*. The operator *T* maps a subset of a Banach space *X* into another Banach space *Y* and *y* is a known element of *Y*. If *y*=0 and *Tx*=*Ux*−*x*, for another operator *U*, the equation *Tx*=*y* is equivalent to the equation *Ux*=*x*. Naturally, to solve *Ux*=*x*, we must assume that the range *R* (*U*) and the domain *D* (*U*) have points in common. Points *x* for which *Ux*=*x* are called fixed points of the operator *U*. In this work, we state the main fixed-point theorems that are most widely used in the field of differential equations. These are the Banach contraction principle, the Schauder–Tychonoff theorem, and the Leray–Schauder theorem. We will only prove the first theorem and then proceed.

Key words: Banach spaces, compact Banach spaces, continuous functions, contraction principle, convex spaces, point-wise convergence, Schauder–Tychonoff
2010 Mathematics Subject Classification: 46B25

INTRODUCTION

Banach contraction principle: Let *X* be a Banach space and *M* be a nonempty closed subset of *X*. Let *T: M*−*M* be an operator such that there exists a constant $k\in$ (0,1) with the property.[1-5]

$$\| Tx - Ty \| \le k \| x - y \| \; for\, every\, x\, y \in M \qquad (1.1)$$

Then, *T* is said to have a unique fixed point in *M*. This result is known as the contraction mapping principle of the fixed-point theorem which we can establish as follows:
Let $x\in M$ be given with $T(x_0)=x_0$, define the sequence $\{x_m\}_0^\infty$ as follows:

$$x_j = Tx_{j-1,} \quad j = 1,2\ldots \qquad (1.2)$$

Then, in William *et al.*'s study (2001), we have $\|x_{j+1}-x_i\|\le k\|_{xj}-x_{j-1}\|\le k^2\|X_{j-1}-x_{j-2}\|\le\in\le k^j\|x_1-x_0\|$
For every $j\ge1$, if $m<n\ge1$, we obtain

$$\begin{aligned}\| x_n, x_n \| &\le \| x_m - x_{m-1} \| + \| x_{m-1} - x_{m-2} \| + \ldots \\ &+ k^n \| x_{n+1} - x_0 \| \\ &\le k^{m-1} \| x_1 - x_0 \| + k^{m-2} \| x_1 - x_0 \| + \ldots \\ &+ k^n \| x_1 - x_0 \| \\ &\le k^n \left(1 + k + k^2 + \ldots + k^{m-n-1}\right) \| x_1 - x_0 \| \\ &\le [k^n / (1-k)] \| x_1 - x_0 \|\end{aligned}$$

Since $k_n\to0$ as $n\to\infty$, it follows that $\{x_m\}_{m=0}^{\infty}$ is a Cauchy sequence. Since *X* is complete, there exists $\bar{x}\in X$ such that $x_m\to\bar{x}$. Obviously, $\bar{x}\in X$ because *M* is closed; taking limits as $j\to\infty$, we obtain $\bar{x}=Tx$. To show uniqueness, let *y* be another fixed point of *T* in *M*. Then,

$$\|\bar{x}-y\|=\|Tx-Ty\|\le k\|\bar{x}-y\|$$

This completes the proof, which implies $\bar{x}=y$ an operator *T. M*−*X*, $M\subset X$, satisfying (1.2) on *M* is called a "contraction operator on *M*."
To illustrate the above due to Kelly (1955) 1.2, let $F: R. XR^n\to R^n$ be is continuous such that

$$\|F(t,x_1)-F(t,x_2)\|\le\lambda(t)\|x_1-x_2\|$$

For every $t\in R_+$, x_1, x_2 $\in R^n$ where $\lambda R_+\to R_+$ is continuous such that

Address for correspondence:
Eziokwu C. Emmanuel,
E-mail: okereemm@yahoo.com

$$\mathrm{L} = \int_0^\infty \lambda(t)\,dt < +\infty$$

Assume further that

$$\int_0^\infty \| F(t,0) \| \,dt < +\infty$$

Then, the operator *T* with

$$(TX)(t) = \int_t^\infty F(s, x(s))\,ds,\ t \in R.$$

maps the space $C_n(R_+)$ into itself and is a contraction operator on $C_n(R_+)$ if $L<1$.

In fact x, y, $C_n^n(R_+)$ be given. Then, due to Bielicki, 1956, we have[6]

$$\begin{aligned} Tx_\infty &\le \int_0^\infty F(t, x(t))\,dt \\ &\le \int_0^\infty F(t, x(t)) - F(t,0)\,dt + \int_0^\infty F\ (t,0)\,dt \\ &\le \int_0^\infty \lambda(t)x(t)\,dt + \int_0^\infty F\ (t,0)\,dt \\ &\le \int_0^\infty \lambda(t)\,dt x_\infty + \int_0^\infty F\ (t,0) \end{aligned}$$

Which shows that $TC_n^n(R_+) \subset C_n^n(R_+)$. We also have

$$\begin{aligned} \| Tx - Ty \|_\infty &\le \int_0^\infty \| F(t, x(t)) - F(t, y(t) \| \,dt \\ &\le \int_0^\infty \lambda(t) \| x(t) - y(t) \| \,dt \\ &\le \| x - y \|_\infty \end{aligned}$$

It follows that if $L<1$, the equation $Tx=x$ has a unique solution $\bar{x}$ in $C_n^n(R_+)$, thus a unique $\bar{x} \in C_n^n(R_+)$ such that

$$\bar{x}(t) = \int_t^\infty F(s, \bar{x}(s))\,ds, t \in R_+$$

It is easily seen that under the above assumptions on *F* and *L* (Banach, 1922), the equation is as follows as:

$$x(t) = f(t) + \int_t^\infty F(s, \bar{x}(s))ds$$

Furthermore, it is a unique solution in $C_n^n(R_+)$ if *f* is a fixed function in $C_n^n(R_+)$. This solution belongs to C_n if $f \epsilon C^n$.

THE SCHAUDER–TYCHONOOFF FIXED-POINT THEOREM

Before we state the Schauder–Tychonoff theorem, we characterize the compact subsets of C^n $[a,b]$.[7] This characterization, which is contained in theorem 2 and 5, allows us to detect the relative compactness of the range of an operator defined on a subset of C^n $[a,b]$ and has values in C^n $[a,b]$. We define below the concept of a relatively compact, and a compact, set in a Banach space.[11-16]

Definition 2.1[8]

Let *X* be a Banach space. Then, a subset *M* of *X* is said to be "compact" if every sequence $\{x_n\}_{n=1}^{\infty}$ in *M* contains a subsequence which converges, i.e., $\{x_n\}_{n=1}^{\infty}$ from *M* contains a subsequence which converges to a vector in *X*.

It is obvious from this definition that is relatively compact if and only if $\bar{M}$ (the closure of *M* in the norm of *X*) is compact. The following theorem characterizes the compact subsets of C^n $[a,b]$.

Theorem 2.1 (H-gham and Taylor, 2003)

Let *M* be a subset of C_n $[a,b]$. Then, *M* is relatively compact if and only if

(i) There exists a constant *K* such that $\|f\|\infty \le K, f \in M$
(ii) The set *M* is "equicontinuous" that is for every $\epsilon>0$ there exists $\delta(\epsilon)>0$ depending only on (ϵ) such that $\|f(t_1)-f(t_2)\|<\epsilon$ for all $t_1, t_2 \in [a,b]$ with $|t_1-t_2|<\delta(\epsilon)$ and all $f \in M$

The proof is based on Lemma 2.3. We start with definition 2.2.

Definition 2.2[9]

Let *M* be a subset of the Banach space *X* and let $\epsilon>0$ be given. Then, the set $M_1 \subset X$ is said to be an "ϵ-net of *M*" if for every point $x \in M_1$ such that $\|x-y\|<\epsilon$.

Lemma 2.2 (Andrzej and Dugundji, 2003)

Let *M* be a subset of a Banach space *X*. Then, *M* is relatively compact if and only if for every $\epsilon>0$ and there exists a finite ϵ net of *M* in *X*.

Proof

Necessity. Assume that M is relatively compact and the condition in the statement of the lemma is not satisfied. Then, there exists some $\epsilon_0>0$, for which there is no finite ϵ_0 net of M, Choose $x_1 \in M$, then, $\{x_1\}$ is not an ϵ_0 net of M. Consequently, $\|x_2 - x_1\| \geq \epsilon_0$ for some $x_2 \in M$. Now consider the set $\{x_2 - x_1\}$. Since this set is not an ϵ_0 net of M, there exists $x_3, \in$, with $\|x_3 - x_1\| \geq \epsilon_0$ for $i=1, 2$. Continuing the same way, we construct as certain any Cauchy sequence, and it follows that no convergent subsequence can be extracted from $\{x_n\}$. This is a contradiction to the compactness of M; thus, for any $\epsilon>0$, there exists a finite ϵ net for M

Sufficiency

Suppose that for every $\epsilon>0$, there exists a finite ϵ net for M and consider a strictly decreasing sequence, $n=1, 2\ldots$ of positive constants such that $\lim_{n\to\infty} \epsilon_n = 0$. Then, for each $n=1, 2\ldots$, there exists a finite ϵ net of M; if we construct open balls with centers at the points of the ϵ_1 net and radii equal to ϵ_1, then every point of M belongs to one of these balls.

Now, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in M. Applying the above argument, there exists a subsequence of $\{x_n\} n=1,2\ldots say \{x_{n_j}\}_{j=1}^{\infty}$ which belongs to one of these ϵ_n- balls. Let B (y_1) be the ball with center y_1. Now, we consider the ϵ_2 net of M. The sequence $\{x'\}$ has now a subsequence $\{x''_n\}, n=1,2\ldots$ which is contained in some ϵ_2- ball. Let us call this ball B (y_2) with center at y_2. Continuing the same way, we obtain a sequence of balls $\{y_n\}_{n=1}^{\infty}$ with centers at y_{n+1}, radii $-\epsilon_n$, and with the following property: The intersection of any finite number of such balls contains a subsequence of $\{x_n\}$. Consequently, choose a subsequence $\{x_{n_k}\}_{k=1}^{\infty} of \{x_n\}$ as follows:

$$x_{n_1} \in B\ (y_1), x_{n_2} \in B(y_2) \cap B(y_1) \ldots x, \in \bigcap_{i=1}^{j} B(y_i)$$

with $n_j > n_{j-1} > \ldots > n_{i-}$

Since x_{nj}, $x_{nk} \in B(y_2)$ for $j \leq k$, we must have

$$\| x_{n_j} - x_{n_k} \| \leq \| x_{n_j} - y_k \| + y_k - x_{n_k} \| < 2_{\delta_k}$$

Thus, $\{x_{nj}\}$ is a Cauchy sequence, and since X is complete, it converges to a point in X. This completes the proof.

Proof of Theorem 2.1

Necessity suffices to give the proof for $n=1$. We assume that M is relatively compact. Lemma 2.1 implies now the existence of a finite ϵ-net of M for any $\epsilon>0$. Let x_1, $x_2(t)$... $x_n(t)$, $t\in$ $[a.b]$ be the function of such an ϵ-net. Then, for every $f\in M$, there exists $x_k(t)$ for which $\|f - x_k\|\infty < \epsilon$ consequently,

$$\begin{aligned} |f(t)| &\leq |x_k(t)| + |f(t) - x(t)|. \\ &\leq \| x_k \|_\infty \| f - x_k \|_\infty \\ &< \| x_k \|_\infty + \in \end{aligned} \tag{2.1}$$

Choose, now, $K = max\|x_k\|\infty + \epsilon$. Since each function $x_k(t)$ is uniformly continuous on $[a,b,]$, there exists $\delta_k(\epsilon)>0$, $k=1,2\ldots$ such that

$$|x_k(t_1)|\infty < \epsilon \text{ for } |t_1 - t_2| < \delta_k(\epsilon)$$

$\delta = \min\ \{\delta_1, \delta_2, \ldots \delta_n\}$ suppose that x is a function in M and let x_j be a function of the ϵ-net for which $\|x - x_j\|\infty < \epsilon$. Then,

$$\begin{aligned} &|x(t_1) - x\ (t_2)| \leq |x(t_1) - x_j(t_1)| + |x_j(t_1) - x_j(t_2)| + \\ &\leq \| x - x_j \|_\infty + |x_j(t_1) - x_j(t_2)| + \\ &< \in + \in + \in = \delta_\in \end{aligned} \tag{2.2}$$

For all t_1, $t_2 \in [a,b]$ with $|t_1 - t_2| < \delta(\epsilon)$. Consequently, M is equicontinuous. The boundedness of M is as follows.

Sufficiency

Fix $\epsilon>0$ and pick $\delta=\delta(\epsilon)>0$ from the condition of equicontinuity. We are going to show the existence of a finite ϵ-net for M Divide $[a,b]$ into subintervals $[t_{k-1}, t_k]$, $k=1,2,\ldots n$ with $t_0=a$, $t_n=b$, and $t_{k-}t_{n-1}<\delta$. Now, define a family P of polygons on $[a,b]$ as follows: the function f: $[a,b]\to[-K,K]$ belongs to P if and only if f is a line segment on $[t_{k-1}, t_k]$ for $k=1,2\ldots$and f is continuous. Thus, if $f\in P$, its vertices (endpoints of its line segments) can appear only at the points $(t_k, f(t_k)$, $k=0,1\ldots n$. It is easy to see that P is a compact set in C_1 $[a,b]$. We show that P is a compact ϵ net of M. To this end, let $t \in [a,b]$. Then, $t \in [t_{j-1}, t_j]$ for some $j=1,2\ldots n$. If Mj and m_j denote the maximum and the minimum of t_{j-1}, t_j, respectively, then

$$m_j \leq x(t) \leq M_j$$
$$m_j \leq \bar{x}(t) \leq M_j$$

Where, $\bar{x}_0$:[a,b]−[K,K] is a polygon in P such that $\bar{x}_0(t_k)=t_k k=1,2\ldots n$. It follows that

$$|x(t)-\bar{x}_0(t)|=\leq M_j-mj<\epsilon$$

Thus, P is a compact ϵ net for M. The reader can now easily check that since P has a finite ϵ-net, say N, the same N will be a finite 2ϵ-net for M. This completes the proof.

The following two examples give relatively compact subsets of functions in C_n [a,b].

Example 2.3: let $M \in C_n^1[a,b]$ be such that there exists positive constants K and L with the property:

i. $\|x(t)\|\leq K$, $t\in [a,b]$ (2.3)

ii. $\|x'(t)\|\leq L$, $t\in [a,b]$

For every $x\in M$, M is a relatively compact subset of $C_n^1[a,b]$; in fact, the equicontinuity of M follows from the Mean Value Theorem for scalar-valued functions.

Proof for Equicontinuity

Consider the operator T of example 1.3.2. Let $M\subset C^n$ [a,b] be such that there exists L>0 with the property:

$$\|x(t)\|\leq L \text{ for all } x\in M$$

Then, the set $S=\{Tu: u\in M\}$ is a relatively compact subset of C^n [a,b]. In fact, if

$$N = \sup_{t\in[a,b]}\int_a^b K(t_1,s) \; ds\,,$$

$\|f\|\leq LN$ for any $f\in S$. Moreover, for $f=Tx$, we have

$$f(t_1)-f(t_2) = \int_a^b\left[K(t_1,s)-K(t_2,s)\right]x(s)ds$$

$$\leq L\int_a^b K(t_1,s)-K(t_2,s)ds$$

This proves the equicontinuity of S

Definition 2.3 (Banach, 1922)

Let X be a Banach space. Let M be a subset of X. Then, M is called "convex" if $\lambda x+(1-\lambda)y\in M$ for any number $\lambda\in [0,1]$ and any $x,y\in M$.

Theorem 2.2 (Schauder–Tychonoff)

Let M be a closed, convex subset of a Banach space X. Let $T: M\to M$ be a continuous operator such that TM is a relatively compact subset of X. Then, T has a fixed point in M.

It should be noted here that the fixed point of T in the above theorem is not necessarily unique. In the proof of the contraction mapping principle, we saw that the unique fixed point of a contraction operator T can be approximated by terms of a sequence $\{x_n\}_{n=0}^{\infty}$ with $x_j = Tx_{j-1}\; j=1,2,\ldots$

Unfortunately, general approximation methods are known for fixed points of operators as in theorem 2.2. which suggests the following definition of a compact operator.

Definition 2.4[10]

Let X, Y be two Banach spaces and M a subset of X. An operator $T: M\to Y$ is called "compact" if it is continuous and maps bounded subsets of into relatively compact subsets of Y.

The example 2.14 below is an application of the Schauder–Tychonoff theorem.

$$<Tx>(t)F(T)+\int_a^b K(t,s)x(s)ds$$

Where, $f\in C^n$ [a,b] is fixed and K: [a,b]x[a,b]$\to M_n$ is continuous. It is easy to show as in examples 1, 3 and 2, 9, that T is continuous on C^n [a,b] and that every bounded set $M\subset C_n$ [a,b] mapped by T onto the set TM is relatively compact. Thus, T is compact. Now let

$M=\{u\in C_n[a,b];\ \|u\|\infty\leq L\}$

Where L is a positive constant. Moreover, let $K+LN\leq L$ where,

$$K = \|f\|_\infty, N = \sup_{t\in[a,b]}\int_a^b \|K(t_1,s)ds$$

Then, M is a closed, convex, and bounded subset of C^n [a,b] such that $TM\subset M$. By the Schauder–Tychonoff theorem, there exists at least one $x_0\in C^n$ [a,b] such that $x_0=Tx_0$. For this x_0, we have

$$x_0(t) = f(t)+\int_a^b K(t,s)x_0(s)ds, t\in[a,b]$$

Corollary 2.4 (**Brouwer's Theorem**: Let $S=\{u\in R^n;\|u\|\leq r\}$

where r is a positive constant. Let T: $S\to S$ be continuous. Then, T has a fixed point in S

Proof

This is a trivial consequence of theorem 2.2 because every continuous function f: $S\to R^n$ is compact.

THE LERAY–SCHAUDER THEOREM

Theorem 3.1 (Leray–Schauder)[8]

Let X be a Banach space and consider the equation.

$$S(x,\mu)-x=0 \quad (3.1)$$

where:

i) $SX\ x\ [0,1]\to X$ is compact in its first variable for each $\mu\in [0,1]$. Furthermore, if M is a bounded subset of X, $S(u,\mu)$ is continuous in μ uniformly with respect to $u\in M$; that is for every $\epsilon>0$, there exists $s\delta(\epsilon)>0$ with the property: $\|S(u,\mu_1)-S(u,\mu_2\|<\epsilon$ for every $\mu_1, \mu_2 \in [0,1]$ with $|\mu_1-\mu_2|>\delta(\epsilon)$ and every $u\in M$
ii) $S(x,\mu_0)=0$ for some $\mu_0\in [0,1]$ and every $x\in X$
iii) If there are any solutions x_μ of the equation (2.4), they belong to some ball of X independently of $\mu\in [0,1]$.

Then, there exists a solution of (3.1) for every $\mu\in [0.1]$,

The main difficulty in applying the above theorem lies in the verification of the uniform boundedness of the solutions (condition (iii). There are no general methods that may be applied to check condition.

As an application of theorem 3.1, we provide example 3.1

Example 3.1 (Arthanasius, 1973)

Let $F: R^n\to R^n$ be continuous and such that for some $r>0$

$$<F(x), x>\leq\|x\|^2 \text{ whenever } \|x\|>r$$

Then, $F(x)$ has at least one fixed point in the ball

$$S_r=[u\in R^n: \|u\|<r]$$

Proof

Consider the equation

$$\mu F(x)-(1+\epsilon)x=0$$

With constants $\mu\in [0,1]$, $\epsilon>0$. Since every continuous function $F: R^n\to R^n$ is compact, the assumptions of theorem 3.1 will be satisfied for (3.1) with $S(x,\mu)=[\mu/((1+\epsilon)]F(x))$ if we show that all possible solutions of (3.1) are in the ball S_r. In fact, let $\bar{x}$ be a solution of (3.1) such that $\|\bar{x}\|>r$. Then, we have

$$<\mu F(\bar{x})-(1+\epsilon)\ \bar{x}, \bar{x}>=0$$

Or

$$<\mu F(\bar{x})\ \bar{x}>=(1+\epsilon)<\bar{x}, \bar{x}>=(1+\epsilon)\|\bar{x}\|^2$$

This implies that

$$<F(\bar{x})\bar{x})>\geq(1+\epsilon)\|\bar{x}\|^2$$

For some $\bar{x}\in R^n$ with $\|\bar{x}\|>r$, which is a contradiction to (3.1).

It follows by theorem 3.1 that for every $\epsilon>0$, the equation (3.1) has a solution x_ϵ for $\mu=1$ such that $\|x_\epsilon\|\leq r$. Let $\epsilon_m=1/(m,m=1,2\ldots)$ and let $x(\in_m)=x_m$ since the sequence $\{x_n\}_{m=1}^{\infty}$ is bounded, it contains a convergent subsequence $\{x_n\}_{k=1}^{\infty}$ let $x_{mk}\to x_0$ as $k\to\infty$ then $x_0\in S_r$ and

$$F\left(x_{m_k}\right)-(1+(1/m_k))x_{m_k}=0, k=1,2\ldots.$$

Taking the limit of the left hand side of the above equation as $k\to\infty$ and using the continuity of F, we obtain

$$F(x_0)=x_0$$

Thus, x_0 is a fixed point of F in S_r.

CONCLUSION

The problem of fixed point is the problem of finding the solution to the equation $y=T_x=0$. It is important that the domain of T and the range of T have points in common, and in this case, such points of x for which $T_x=x$ are regarded as the fined points of the operator T; also, the work reveals that the contraction mapping principle must be satisfied for a fixed point to exist as other basic results center on the need for the relative compactness of a subset, M of $C_n\ [a,b]$ if there must be a fixed point of T in M.

REFERENCES

1. Grannas A, Dugundji J. Fixed Point Theory. New York: Publish in Springer-Verlag; 2003.
2. Kartsatos AG. Advanced Ordinary Differential Equations. New York: Hindawi Publishing Corporation; 1973.
3. Banach S. On operations Danes sets abstracts and their application to integral equations. Fundam Math 1922;3:133-81.
4. Banach S. On linear functionals II. Studia Math 1929;1:223-39.
5. Banach S. Theories Des Operations Lineaires. New York: Chelsea; 1932.
6. Bielicki A. A remark on the banachment method Cacciopoli-Tikhonov. Bull Acad Pol Sci 1956;4:261-8.
7. H-gham DJ, Taylor A. The Sleckest Link Algorithin; 2003. Available from: http://www.maths.strath.AC.uk/<<aas96106/rep/20. [Last accessed on 2018 Dec 06].
8. Day MM. Normed Linear Spaces. 3rd ed. New York: Springer: 1973.
9. Hille E, Philips RS. Functional Analysis and Semi Groups. 31st ed. Providence, RI: American Mathematical Society; 1957.

10. Holder O. Ubeer an average value set. Nachr Akad Wiss Gottingen Math Phys Kl 1889;12:38-47.
11. Ince EL. Ordinary Differential Equations. New York: Dover; 1956.
12. Kelley JL. General Topology. New York: Van Nostrand; 1955.
13. Willians AK. Mohamed AK. An Introduction to Metric Spael and Fixed Point Theorem. New York: John Willey; 2001.
14. Todd J. Survey of Numerical Analysis. New York: McGraw-Hill; 1962.
15. Willian A, Sims KB. Hand Book of Metric Fixed Point Theorem. London: Kluwer Academy; 2001.
16. Yosida K. Fundamental Analysis. 3rd ed. Berlin: Springer; 1971.