ISSN 2581-3463

AJMS

Asian Journal of Mathematical Sciences

RESEARCH ARTICLE

On Application of Power Series Solution of Bessel Problems to the Problems of Struts of Variable Moments of Inertia

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Received: 25-06-2019; Revised: 15-08-2019; Accepted: 17-10-2019

ABSTRACT

One of the most powerful techniques available for studying functions defined by differential equations is to produce power series expansions of their solutions when such expansions exist. This is the technique I now investigated, in particular, its feasibility in the solution of an engineering problem known as the problem of strut of variable moment of inertia. In this work, I explored the basic theory of the Bessel's function and its power series solution. Then, a model of the problem of strut of variable moment of inertia was developed into a differential equation of the Bessel's form, and finally, the Bessel's equation so formed was solved and result obtained.

Key words: Bessel's equations, power series singular point, regular points, strut of variable moment of inertion

2010 Mathematics Subject Classification: 33C10

INTRODUCTION

The heart of this work is centered on the Bessel's equations of the ordinary differential equation, the solution of which I approached only through the power series method. Hence, I discussed the basics of the theory of power series solutions of ordinary differential equations and Bessel's equation in this section.

Basics on the power series

We now begin by recalling some basic facts about power series.

I. An expression Bayin[1] of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^z + \ldots \tag{1.1}$$

In which the a_n is constants, is called a power series in x while

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n . \tag{1.2}$$

In which, x_n is also a constant, is called a power series in $x - x_0$, since Equation (1.2) can always be transformed into Equation (1.1) by the change of variable $u = x - x_0$. It is important to note that,

II. A power series Bayin[2] in x is said to converge when $x = x_1$ if

$$\sum_{n=0}^{\infty} a_n x_1^n \tag{1.3}$$

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is a convergent series of real numbers, in which case Equation (1.3) is called the sum of the series at x_1. Otherwise, the series is said to diverge at x_1.

Every power series in x obviously converges when $x = 0$, and its sum at that point is a_0, the constant term of the series. More generally, every power series has an associated radius of convergence R,. where $0 \le R \le \infty$, which is characterized by the property that the series converges when $|x| < R$ and diverges when $|x| > R$. In other words,

$$\sum_{n=0}^{\infty} a_n x^n$$

converges inside an interval R centered at 0 and diverges outside that interval. (Convergence or divergence at the endpoint of the interval must be determined on a case-by-case basis by examining the particular series in question.)

III. The radius of convergence of many power series Bentely and Cooke[3] can be found by means of the ratio test:

If

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1} x_1^{n+1}}{a_n x_1^n} \right| = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| |x_1|,$$

Then

$$\sum_{n=0}^{\infty} a_n x_1^n$$

converges when $L < 1$ and diverges when $L > 1$.

This test immediately implies, for instance, that the series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots,$$

converges for $|x| < 1$ and diverges for $|x| > 1$. Hence, its radius of convergence is 1.

IV. A power series Brand[4] in x with a positive radius of convergence R defines a function f in the interval $|x| < R$ by the rule.

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{n=0}^{\infty} a_n x^n \tag{1.4}$$

This function is continuous and has derivatives of all orders everywhere in the interval. Moreover, these derivatives can be found by differentiating Equation (1.4) term by term.

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots = \sum_{n=0}^{\infty} n a_n x^{n-1} \tag{1.5}$$

$$f''(x) = 2a_2 + 3.2a_3 x + \ldots = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}.$$

And so on.

V. A function f that can be represented by a convergent power series Brauer and Nohel[5] of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

for all x in an open interval I centered at x_0 is said to be analytic at x_0. In this case, the coefficients of the series are uniquely determined by the formula

$$a_n = \frac{f^{(n)}(x_0)}{n!},$$

Where $f^{(n)}(x_0)$ denotes the n^{th} derivative of f evaluated at x_0. In particular, if $f(x)=0$ for all x in I then $a_n=0$ for all n. Thus, if

$$\sum_{n=0}^{\infty} a_n\,(x-x_0)^n=\sum_{n=0}^{\infty} b_n(x-x_0)^n$$

for all x in I then $a_n=b_n$ for all n because

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n-\sum_{n=0}^{\infty} b_n(x-x_0)^n=\sum_{n=0}^{\infty}\left(a_n=b_n\right)(x-x_0)^n=0$$

for all x in I. We shall use these facts repeatedly, though usually without explicit mention, in our work with differential equations.

It turns out that if f is analytic at x_0, it is actually analytic at each point in its interval of convergence about x_0. Thus, it is customary to speak of functions as being analytic on an interval, the phrase "analytic at x_0" being used only to direct attention to the point about which the series is expanded.

VI. Every polynomial in one variable Coddington and Levinson[6] is analytic on the entire real line, since

$$a_0+a_1x+\ldots+a_kx^k$$

can be viewed as a power series in x with $a_n=0$ for $n>k$. In fact, the notion of an analytic function can be seen as a generalization of the notion of a polynomial, and these two classes of functions have many properties in common. For instance, both are vector spaces n which addition and scalar multiplication are performed term by term, and in both there is a well-defined multiplication. Thus, if f and g are analytic on an open interval I, then so are $f+g, \propto f$ for any scalar $\propto$, and fg. Moreover, if

$$f(x)=a_0+a_1x+\ldots=\sum_{n=0}^{\infty} a_nx^n$$

Moreover,

$$g(x)=b_0+b_1x+\ldots=\sum_{n=0}^{\infty} b_nx^n$$

Then, fg is computed according to the formula

$$(fg)(x)=a_0b_1+\left(a_0b_1+a_1b_0\right)x+\left(a_0b_2+a_1b_1+a_2b_0\right)x^2+\ldots$$

In other words,

$$f(x)=\sum_{n=0}^{\infty} a_nx^n \text{where}$$

$$c_n=a_0b_n+a_1b_{n-1}+\ldots+a_nb_0.$$

VII. We Coddington[7] have already observed that polynomials are analytic on the entire real line. So are the functions e^x, $\sin x$ and $\cos x$ and their power series expansions about $x=0$ are

$$e^x=1+x+\frac{x^2}{2!}+\ldots=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!'}$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots=\sum_{n=0}^{\infty}(1-)^n\frac{x^{2n}}{(2n)!}.$$

These series are encountered so frequently that it is well worth the effort to remember them.

Rational function (quotients of polynomials) is also analytic wherever they are defined. Thus,

$$\frac{p(x)}{q(x)}$$

Is analytic at $x=0$ when p and q are polynomials and $q(0)\neq 0$.

$$\sum_{n=0}^{\infty} a_n x^n \tag{1.6}$$

is a "dummy variable" an can be changed whenever it is convenient to do so. For instance, if we replace n by $n+1$ in Equation (1.6), we obtain

$$\sum_{n+1=0}^{\infty} a_{n+1} x^{n+1}.$$

which can be rewritten as

$$\sum_{n=-1}^{\infty} a_{n+1} x^{n+1}$$

This substitution has the effect of changing the index of summation in the original series by one. In our work with differential equations, we shall use this maneuver to rewrite the formulas for the first and second derivatives of Equation (1.6) as

$$f'(x)=\sum_{n=0}^{\infty}(n+1)a_{n-1}x^n$$

and

$$f''(x)=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n-2}x^n.$$

Series solution about an ordinary point

We now resume the study of the equation

$$p(x)\frac{d^2y}{dx^2}+q(x)\frac{dy}{dx}+r(x)y=0 \tag{1.7}$$

and impose the restriction that $p, q,$ and r be analytic on an open interval I of the $x-$axis. As we shall see, the behavior of the solution of Equation (1.7) in a neighborhood of a point x_0 in I depends in large measure on whether $p(x_0)=0$ or not. In the former case x_0 in I depends in large measure on whether $p(x_0)=0$ or not. In the former case, x_0 is said to be a singular point for the equation; in the later case, it is said to be an ordinary point. We begin by considering solution about ordinary points, the easier of the two cases.

When $p(x_0)\neq 0$, the continuity of p implies the existence of an interval about x_0 in which $p(x)\neq 0$. Thus, if we restrict our attention to that interval. Equation (1.7) can be rewritten Hurewiez[8] as.

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0 \tag{1.8}$$

Were $P(x)=q(x)/p(x)$ and $Q(x)=r(x)/p(x)$. In this form, x_0 is an ordinary point for the equation if P and Q are analytic in an interval about x_0. The following theorem, which we state without proof, describes the solution of Equation (1.8) in this case.

Theorem 1.1 Ince[9] if the coefficients P and Q in the equation

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$$

are analytic at x_0 and have power series expansions that converge in the interval verify $|x-x_0|<R$, then every solution of the equation is analytic at x_0 and its power series expansion also converges when $|x-x_0|<R$.

Theorem (1.1) states that about x_0, the solution of

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$$

can be written in the form

$$y=\sum_{n=0}^{\infty}a_n(x-x_0)^n.$$

where they a_0 are constants.

Example 2 (Baver and Nohel (1967)) find the general solution of

$$y''+xy'+y=0 \tag{1.13}$$

Solution: Theorem 1.1 again guarantees that the solution of this equation has power series expansions of the form

$$y=\sum_{n=0}^{\infty}a_nx^n$$

That converges for all values of x. We now substitute this series and its first two derivatives in the differential equation to obtain.

$$\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}na_nx^n+\sum_{n=2}^{\infty}a_nx^n=0.$$

Next, we shift the index of summation in the first series and collect like terms to obtain first

$$\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}na_nx^n+\sum_{n=0}^{\infty}a_nx^n=0,$$

And then,

$$a_0+2a_2+\sum_{n=1}^{\infty}n+1\left[(n+2)a_{n+2}+a_n\right]x^n=0.$$

Thus,

$$a_0+2a_2=0$$

$$(n+2)a_{n+2}+a_n=0,\quad n\geq 1,$$

Moreover, we are again in a position to determine all the a_n in terms of a_0 and a_1. They are as follows:

n even	n odd
a_0	a_1
$a_2=-\frac{a_0}{2}$	$a_3=-\frac{a_1}{3}$
$a_4=-\frac{a_2}{4}=\frac{a_0}{2.4}$	$a_5=-\frac{a_3}{5}=\frac{a_1}{3.5}$
$\vdots$	$\vdots$
$a_{2n}=(-1)^n\frac{a_0}{2.4\ldots(2n)}$	$a_{2n-1}=(-1)^n\frac{a_1}{3.5\ldots(2n+1)}$

Thus,

$$y=a_0\left(1-\frac{x^2}{2}+\frac{x^4}{2.4}-\ldots\right)+a_1\left(x-\frac{x^3}{3}+\frac{x^5}{3.5}-\ldots\right)$$

$$=a_0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^nn!}+a_1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n-1}}{1.3\ldots(2n+1)'}$$

Where a_0 and a_1 are arbitrary constants, and the equation has been "solved."

In addition, the power series expansion of a solution of a differential equation often reveals properties of the solution that would be exceedingly difficult to discover by other means. For this reason, an analysis of the functions defined by a differential equation usually begins with an attempt to produce a power series expansion of those functions. At the same time, however, we should point out that power series solutions are not always as easy to come by as the examples we have given might suggest. The difficulty arises in connection with the recurrence relation which can easily be too complicated to yield a formula for the coefficients of the series. Examples are given in the exercises that follow.

Singular points

We (Kamaziha and Prudunkov (2001)) recall that x_0 is a singular point for the equation

$$p(x)\frac{d^2y}{dx^2}+q(x)\frac{dy}{dx}+r(x)y=0 \tag{1.14}$$

If p, q and r are analytic at x_0 and $p(x_0)=0$. In general, there is very little that can be said about the nature of the solution of a differential equation about a singular point. There is, however, one important exception, namely, the case where x_0 is a "regular" singular point in the sense of the following definition.

$$(x-x_0)^2\frac{d^2y}{dx^2}+(x-x_0)P(x)\frac{dy}{dx}xQ(x)y=0, \tag{1.15}$$

Where P and Q are analytic at x_0. A singular point that is not regular is said to be irregular. In the following discussion, we shall limit ourselves to equations that have a regular singular point at the origin in which case Equation (1.15) becomes

$$x^2y''+xP(x)y'+Q(x)y=0 \tag{1.16}$$

As we observed earlier, this limitation involves no loss of generality since the change of variable $u=x-x_0$ will move a singularity form x_0 to 0.

Example 2: Kamazina and Prudinkov[10] find the general solution of

$$x^2y''+x(x-\frac{1}{2})y=0 \tag{1.17}$$

On $(0,\infty)$ and $(-\infty,0)$,

Solution: We begin by considering the interval $\propto<0$ where we seek a solution of the form

$$y=x^v\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}a_nx^{n+v}, \tag{1.18}$$

Where $a_0\neq 0$ and v are arbitrary. (This particular guess as to the form of a solution is motivated by our study of series solution and the results obtained for the Euler equation). Then,

$$y'=\sum_{n=0}^{\infty}(n+v)a_nx^{n+v-1}=x^v\sum_{n=0}^{\infty}(n+v)a_nx^{n-1},$$

$$y''=\sum_{n=0}^{\infty}(n+v)(n+v-1)a_nx^{n-v-2}$$

$$=x^v\sum_{n=0}^{\infty}(n+v)(n+v-1)a_nx^{n-2}$$

Moreover, Equation (1.17) implies that

$$\sum_{n=0}^{\infty}(n+v)(n+v-1)a_nx^n+\sum_{n=0}^{\infty}(n+v)a_nx^{n+1}-\frac{1}{2}\sum_{n=0}^{\infty}(n+v)a_nx^n+\frac{1}{2}\sum_{n=0}^{\infty}a_nx^n=0$$

However, since

$$\sum_{n=0}^{\infty}(n+v)a_nx^{n+1}=\sum_{n=1}^{\infty}(n+v-1)a_{n-1}x^n,$$

The preceding expression can be rewritten as

$$\left[v(v-1)-\frac{1}{2}v+\frac{1}{2}\right]a_0+\sum_{n=1}^{\infty}[(n+v)(n=v-1)-\frac{1}{2}(n+v)+\frac{1}{2}]a_nx^n+\sum_{n=1}^{\infty}(n+v-1)a_{n-1}x^n=0$$

Thus, by assumption $a_0=0$ and

$$[n+v)(n+v-1)-\frac{1}{2}(n-v)+\frac{1}{2}]a_n-(nv-1)a_{n-1}=0 \quad (1.19)$$

Equation (1.18) determines the admissible value of v for this problem as $\frac{1}{2}$ and 1 is known as the indicial equation associated with Equation (1.15) substituting these values in Equation (1.19) we find that when $v=\frac{1}{2}$.

$$a_n=\frac{a_{n-1}}{n}.$$

Moreover, when $v=1$,

$$a_n=-\frac{2}{2n+1}a_{n-1}$$

From these, we obtain the following two sets of values for the a_n both expressed in terms of a_0:

$\underline{v=\frac{1}{2}}$	$\underline{v=1}$
$a_1=-a_0$	$a_1=-\frac{2}{3}a_0$
$a_2=-\frac{a_0}{2!}$	$a_2=-\frac{2^3}{3.5}a_0$
$a_3=-\frac{a_0}{3!}$	$a_3=-\frac{2^3}{3\,.\,5\,.\,7}a_0$
$\vdots$	$\vdots$

finally setting, $a_0=1$ we conclude that each of the following series formally satisfies Equation (1.13).

$$y_1=x^{1/2}\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots\right)$$

$$y_2=x\left(1-\frac{2x}{3}+\frac{(2x)^2}{3.5}-\frac{(2x)^3}{3\,.5\,.7}+\ldots\right)$$

At this point, we have proved that if

$$x^2y''+x\,(x=-\frac{1}{2})y'+\frac{1}{2}y=0$$

Has a solution of the form

$$y=x^v\sum_{n=0}^{\infty}a_nx^n,$$

With $a_0=1$ then, this solution must be one of the two series found above. As yet, however, we have no guarantee that either of these series actually is a solution of the given equation, since it is conceivable that both of them might diverge for all $x>0$. This is what we meant a moment ago when we said that these series "formally" satisfy Equation (1.13). Fortunately, an easy computation by the ratio test disposes of this difficulty. Both series converge for all $x>0$, and since y_1 and y_2 are also linearly independent in C (0, ∞) the general solution of Equation (1.13) for $x>0$ is

$$y=c_1y_1+c_2y_2,$$

Where c_1 and c_2 are arbitrary constants?

Finally, to remove the restriction on the interval, we observe that the preceding argument still holds if we replace x^v by $|x|^v$ throughout; that is by $(-x)^v$ for $x<0$. Thus, the general solution of Equation (1.13) on any interval not containing the origin is

$$y=c_1|x|^{1/2}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}+c_2|x|\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^n}{1.3\ .5\ldots(2n+1)}.$$

In Equation (1.13) and we deduce that the general solution of the equation on $(-\infty,0)$ is

$$y=c_1|x|^{1/2}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}+c_2|x|\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^n}{1\ .3\ldots(2n+1)}.$$

Solution about a regular singular point

In this section, we Kaplan[11] shall indicate how the method of Frobenius can always be used to obtain at least one solution of

$$x^2y''+xP(x)y'+Q\ (X)\,y=0 \tag{1.20}$$

About $x=0$ whenever P and Q are analytic at that point. Once again we begin by letting x be positive, in which case we seek a solution of Equation (1.20) of the form

$$y=x^v\sum_{n=0}^{\infty}a_nx^n \tag{1.21}$$

With $a_0\neq 0$ then

$$y=x^v\sum_{n=0}^{\infty}(n+v)\,a_nx^{n-1}$$

$$y=x^v\sum_{n=0}^{\infty}(n+v)(n+v-1)\,a_nx^{n-2},$$

Moreover, Equation (1.21) yields

$$\sum_{n=0}^{\infty}(n+v)(n+v-1)a_nx^n+P\ (x)\sum_{n=0}^{\infty}(n+v)\,a_nx^n$$
$$+Q(x)\sum_{n=0}^{\infty}a_nx^n=0\ . \tag{1.22}$$

However, since P and Q are analytic at $x=0$ it follows that

$$P(x)=\sum_{n=0}^{\infty}p_nx^n \text{ and } Q(x)=\sum_{n=0}^{\infty}q_nx^n$$

Where both series converge in an interval $|x|<R, R>0$. Thus, Equation (1.22) can be rewritten as

$$\sum_{n=0}^{\infty}(n+v)(n+v-1)\,a_nx^n+\left(\sum_{n=0}^{\infty}(n+v)\,a_nx^n\right)\left(\sum_{n=0}^{\infty}p_nx^n\right)$$
$$+\left(\sum_{n=0}^{\infty}a_nx^n\right)\left(\sum_{n=0}^{\infty}q_nx^n\right)=0$$

Moreover, if we now carry out the indicated multiplication according to the formula given on p.268 we obtain.

$$\sum_{n=0}^{\infty}\left[(n+v)(n+v-1)\,a_n+\sum_{j=0}^{n}j+\sum_{j=0}^{n}a_jq_{n-j}\right]x^n=0.$$

Hence, Equation (1.22) will satisfy Equation (1.21) formally in the interval $0 < x < R$ if

$$(n+v)(n+v-1)a_n + \sum_{j=0}^{n}[(j+v)p_{n-j} + q_{n-j}]a_j = 0 \tag{1.23}$$

For all $n \geq 0$.

When $n = 0$ Equation (1.23) reduces to

$$v(v-1) + p_0(n+v) + q_0 = 0. \tag{1.24}$$

Moreover, when $n \geq 1$ it can be written as

$$[(n+v)(n+v-1) + p_0(n+v) + q_0]a_n + \sum_{j=0}^{n-1}\left[(j+v)p_{n-j}\right]a_j = 0. \tag{1.25}$$

The first of these relationships is known as the indicial equation associated with Equation (1.20) its roots determine the admissible values of v in Equation (1.21). Notice that since p_0 and q_0 are the constant terms in the series expansions of P and Q Equation (1.24) may be rewritten as.

$$v(v-1) + P(0)v + Q(0) = 0 \tag{1.26}$$

Thus, when P and Q are explicitly given, the indicial equation associated with Equation (1.20) can be written down at sight. To continue we set.

$$I(v) = v(v-1) + p_0 v + q_0,$$

Moreover, let v_1 and v_2 denote the roots of the equation $I(v) = 0$. Moreover, we suppose that v_1 and v_2 have been labeled so that $Re(v_1) \geq Re(v_2)$* then when $v = v_1$,

By $Re(v)$, we mean the real part of the complex v. Thus, if $v = \propto + \beta_i$ $Re(v) = \propto$ the real part of a number is, of course, the number itself Equation (1.24) becomes.

$$I(n+v_1)a_n + \sum_{j=0}^{n-1}\left[(j+v_1)p_{n-j} + q_{n-j}\right]a_j = 0 \, n \geq 1, \tag{1.27}$$

Moreover, since the choice of v_1 implies that $\{(n+v_1) \neq 0 \text{ for } n > 0$ we can solve Equation (1.27) for a_n to obtain

$$I(n+v_2)a_n + \sum_{j=0}^{n-1}\left[(j+v_1)p_{n-j} + q_{n-j}\right]a_j, \quad n \geq 1.$$

This relation determines all the a_n from $n = 1$ onward in terms of a_0 and yields a formal solution of Equation (1.21) in the interval $0 < x < R$. Moreover, if x^v is replaced by $|x|^v$ throughout these computations, we obtain a formal solution in the interval $-R < x > 0$. Finally, the resulting series is known to converge when $0 < |x| < R$, and is, therefore, a solution of Equation (1.13).*

To complete the discussion, we Kreyszig[12] must now find a second solution of Equation (1.23) that is linearly independent of the one just obtained. We attempt to repeat the preceding argument using the second root v_2 of the indicial equation. If $v_2 = v_1$, we get nothing new of course new. However, if $v_2 \neq v_1$

Equation (1.27) becomes

$$I(n+v_2)a_n + \sum_{j=0}^{n-1}\left[(j+v)p_{n-j} + q_{n-j}\right]a_j = 0.$$

A can again be solved for a_n provided

$$I(n+v_2) \neq 0$$

For all $n < 0$. However, when $n > 0, I(n+v_2) = 0$ if and only if $n + v_2 = v_1$ that is if and only if $v_1 - v_2 = n$. Thus, our technique will yield a second solution of Equation (1.21) for $0 < |x| < R$ whenever the roots of the indicial equation $I(v) = 0$ do not differ by an integer. In this case, it is easy to show that the (particular) solution $y_1$and y_2 obtained by setting $a_0 = 1$ in these series are linearly independent and hence that the general solution of Equation (1.27) about the origin is

$y = c_1 y_{1+} + c_2 y_2$

Where c_1 and c_2 are arbitrary constants? This, for instance, is what happened in the example in the preceding section. In the next section, we shall discuss the so-called exceptional cases in which $v_2 – v_1$ is an integer.

Solution about a regular singular point: The exception cases

To complete the discussion of solution about a regular singular point, it remains to consider the case where v_1 and v_2 the roots of the indicial equation differ by an integer. Our experience with the Euler equation suggests that a solution involving a logarithmic term should arise when $v_1 = v_2$ and, as we shall see, this can also happen when $v_1 \neq v_2$. The following theorem gives a complete description of the situation, both for the case already treated and for each of the exceptional cases.

Theorem 1.4 Leighton[13] let

$$x^2 y'' + xP(x)y' + Q(x)y = 0 \quad (1.28)$$

Be a second-order homogeneous linear differential equation whose coefficients are analytic in the interval $|x| < R.R > 0$, and let v_1 and v_2 be the roots of the equation

$v(v-1) + P(0)v + Q(0) = 0,$

Where v_1 and v_2 are labeled so that $Re(v_1) \geq Re(v_2)$. Then, Equation (1.28) has two linearly independent solutions y_1 and y_2 valid for $0 < |x| < R$, whose form depends on v_1 and v_2 in the following way.

Case 1. $v_1 - v_2$ not an integer.

$$y_1 = |x|^{v_1} \sum_{n=0}^{\infty} a_n x^n, \quad a_0 \neq 0,$$

$$y_2 = |x|^{v_2} \sum_{n=0}^{\infty} b_n x^n, b_0 \neq 0,$$

Case 2. $v_1 = v_2 = v.$

$$y_1 = |x|^{v} \sum_{n=0}^{\infty} a_n x^n, \quad a_0 \neq 0,$$

$$y_2 = |x|^{v} \sum_{n=1}^{\infty} b_n x^n + y_1(x) \text{ in } |x|.$$

Case 3. $v_1 - v_2\, a$ positive integer,

$$y_1 = |x|^{v_1} \sum_{n=0}^{\infty} a_n x^n, \quad a_0 \neq 0,$$

$$y_2 = |x|^{v_2} \sum_{n=0}^{\infty} b_n x^n - c y_1(x) \text{in} |x| b_0 \neq 0 . c\, a (\text{fixed}) \text{constant.*}$$

Note that, when $c = 0$ does not contain a logarithmic term.

Finally, the values of the coefficients in each of these series are uniquely determined up to an arbitrary constant and can be found directly from the differential equation by the method of undetermined coefficients.

We shall not attempt to prove this theorem but shall instead present an argument that suggests why a solution involving a logarithmic term arises when $v_1 = v_2$. The argument goes as follows. As before, we begin by attempting to determine the a_n in

$$x^v \sum_{n=0}^{\infty} a_n x^n$$

So that the resulting expression satisfies Equation (1. 28) for $0 < x < R$. This time, however, we also regard v as a variable and set

$$y(x,v) = x^v \sum_{n=0}^{\infty} a_n x^n \tag{1.29}$$

Moreover, we assume from the outset that $a_0 = 1$. Then, if L denotes the linear differential operator $x^2D^2 + xP(x)D + Q(x)$, our earlier discussion implies that.

$$Ly(x,v) = I(v)x^v + x^v \sum_{n=1}^{\infty} \left\{ I(n+v)a_n + \sum_{j=0}^{n-1} \left[(j+v) p_{n-j} + q_{n-j} \right] a_j \right\} x^n, \tag{1.30}$$

Where they p_{n-j} and q_{n-j} are the coefficients of the power series expansions of P and Q about the origin. We now use the recurrence relation.

$$I(n+v)a_n + \sum_{j=0}^{n-1} \left[(j+v) p_{n-j} + q_{n-j} \right] a_j = 0$$

To determine $a_1, a_2 ...$ in terms of v so that every term but the first on the right-hand side of Equation (1.30) vanishes. If the resulting expression is denoted by $a_n(v)$ and substituted in Equation (1.29) we obtain a function

$$y_1(x,v) = x^v \left[1 + \sum_{n=0}^{\infty} a_n \ (v) x^n \right] \tag{1.31}$$

With the property that

$$Ly_1 = I(v)x^v. \tag{1.32}$$

However, v_1 is a double root of the equation $I(v) = 0$. Therefore, $I(v) = (v - v_1)^2$ and

$$Ly_1 = (v - v_1)^2 x^v. \tag{1.33}$$

Thus, $Ly_1 = 0$ when $v = v_1$, and the expression $y_1(x, v_1)$ formally satisfies the equation $Ly = 0$. This, of course, agrees with our earlier results.

The idea behind obtaining the second solution, in this case, originates with the observation that when Equation (1.33) is differentiated with respect to v its right-hand side still vanishes when $v = v_1$. Indeed,

$$\frac{\hat{c}}{\hat{c}v} (v - v_1)^2 x^v (v - v_1)[2 + (v - v_1) \text{In}\, x]$$

However, since

$$\frac{\hat{c}}{\hat{c}v} \left[Ly_1(x,v) \right] = L\left[\frac{\hat{c}}{\hat{c}v} y_1(x,v) \right],$$

Equation (1.33) implies that

$$L\left[\frac{\hat{c}}{\hat{c}v} y_1(x,v) \right] = 0$$

When $v = v_1$ thus, if we differentiate Equation (1.31) term by term with respect to v and then set $v = v_1$ the resulting expression will also formally satisfy the equation $Ly = 0$ when $0 < x < R$. Denoting this expression by $y_2(x, v_1)$ we have

$$y_1(x, v_1) = \frac{\partial}{\partial v} y_1 \ (x,v)\Big|_{v=v_1}$$

$$= x^{v_1} \sum_{n=1}^{\infty} a'_n (v_1) x^n + y_1(x, v_1) \text{In } x$$

This is precisely the form of the second solution given in the statement of theorem 4.1 under Case 2.

BESSEL'S EQUATION

The differential equation

$$x^2y''+xy'+(x^2-p^2)y=0 \tag{2.1}$$

Where p is a non-negative real number, is known as Bessel's equation of order p. It is one of the most important differentials in applied mathematics, and as a consequence its solution, which is called Bessel function, has been intensively studied. In this section and the next, we shall derive some of the more elementary properties of these function, including their expansions about the regular singular point at $x=0$. For simplicity, we shall confine our attention to the nonnegative $x-$axis. The indicial associated with Equation (2.1) is

$$v^2-p^2=0$$

Which has the roots $\pm p$. Thus, Bessel's equation of order p has a solution of the form

$$y=x^p\sum_{k=0}^{\infty}a_kx^k,$$

With $a_0\neq 0.*$ to evaluate the a_k in this series, we Rabinstein (1972) observe that

$$\left(x^2-p^2\right)y=x^p\sum_{k=2}^{\infty}a_{k-2}x^k-x^p\sum_{k=0}^{\infty}p^2a_kx^k$$

$$xy'=x^p\sum_{k=0}^{\infty}\left(k+p\right)\left(k+p-1\right)a_kx^k$$

$$x^2y''=x^p\sum_{k=0}^{\infty}\left(k+p\right)\left(k+p-1\right)a_kx^k.$$

When these expressions are substitution in Bessel's equation we find that

$$\sum_{k=0}^{\infty}\left[\left(k+p\right)\left(k+p-1\right)+\left(k+p\right)-p^2\right]a_kx^k+\sum_{k=2}^{\infty}a_{k-2}x^k=0$$

Because it is traditional to use the letter n rather than p when Bessel's equation is of integral order, we shall use k as the index of summation throughout this discussion.

Or

$$\left(2p+1\right)a_1x+\sum_{k=2}^{\infty}\left[k\left(2p+k\right)a_k+a_{k-2}\right]x^k=0.$$

From this follows that

$$a_1=0 \text{ and } a_k=-\frac{a_{k-2}}{k(2p+k)},\qquad k\geq 2.$$

Thus,

$$a_1=a_3=a_5=\ldots=0,$$

While

$$a_2=-\frac{a_0}{2(2p+2)},$$

$$a_4=\frac{a_0}{2\,.4\,.\left(2p+2\right)(2p+4)},$$

$$\vdots$$

$$a_{2k}=(-1)^k\frac{a_0}{2\,.4\ldots\left(2k\right)\left(2p+2\right)\left(2p+4\right)\ldots\left(2p+2k\right)}$$

$$=(-1)^k\frac{a_0}{2^{2k}\,k!\left(p+1\right)\left(p+2\right)\ldots\left(p+k\right)}$$

Hence,

$$y = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+p}}{2^{2k}\, k!(p+1)(p+2)\ldots(p+k)}, \tag{2.4}$$

Where, $a_0 \neq 0$ is an arbitrary constant? From this point onward, the discussion divides into cases that depend on the value of p

Bessel function of the first kind

1. $p = n,$ an integer. In this case, Equation (2.2) assumes a particularly simple form when we set.

$$a_0 = \frac{1}{2^n n!} \tag{2.3}$$

The corresponding solution of Bessel's equation is denoted by $J_n(x)$ and is called the Bessel function of order n of the first kind:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{2k+n} \tag{2.4}$$

In particular, Leighton[13] the series expansion of J_0, with J_1 is the most important of the Bessel function is

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k}$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^2}{2^2\,.4^2} - \frac{x^6}{2^2\,.4^2\,.6^2} + \ldots \tag{2.5}$$

Graphs $J_0.J_1.$and J_2 are shown in Figure 1 note the oscillatory behavior of these functions. This phenomenon will be discussed in some detail in the next chapter.

2. p not an integer; the gamma function. To obtain a formula for J_p when p is not an integer that is analogous to the formula, for J_n, we must generalize the notion of the factorial function to include nonintegral values of its argument. Such a generalization was originally discovered by Euler and is known as the gamma function. It is defined by the improper integral Lizorkin[14].

$$I(p) = \int_0^{\infty} t^{p-1} e^{-1}\, dt, \qquad p > 0, \tag{2.6}$$

Which converges for all $p > 0$.

The fact that $\Gamma(p)$ generalizes the factorial function is a consequence of the functional equation,

$$\Gamma(p+1) = p\Gamma(p). \tag{2.7}$$

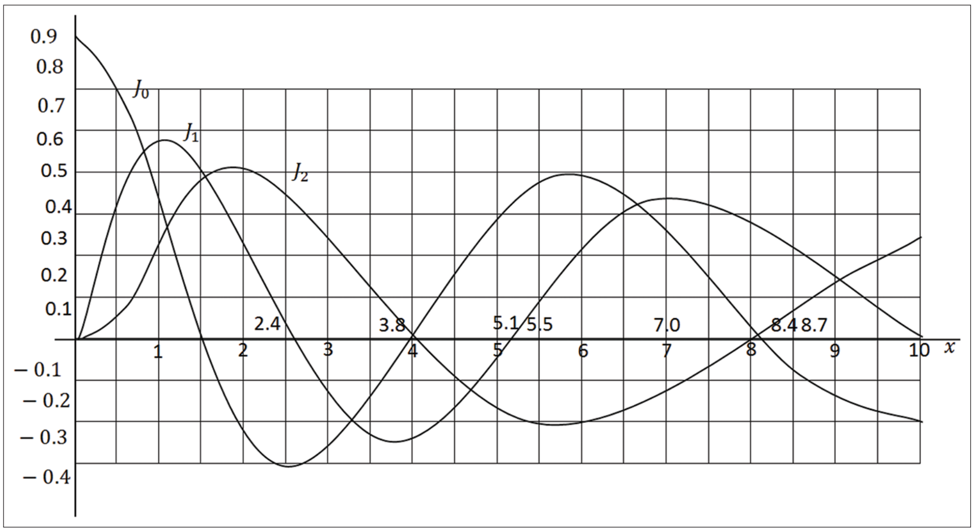

Figure 1: Illustrating Interlacing of the zeros of the J-Bessel Functions

To prove Equation (2.7), we use integration by parts to evaluate $\Gamma(p+1)$ as follows:

$$\Gamma(p+1) = \int_0^\infty t^p e^{-1}\,dt$$

$$= -t^p e^{-1}\Big|_0^\infty + p\int_0^\infty t^{p-1}e^{-1}\,dt$$

$$= p\int_0^\infty t^{p-1}e^{-t}\,dt$$

$$p\Gamma(p).$$

Thus, since

$$\Gamma(1) = \int_0^\infty e^{-1}dt = 1$$

We have

$$\Gamma(2) = 1\Gamma(1) = 1,$$
$$\Gamma(3) = 2\Gamma(2) = 2.1,$$
$$\Gamma(4) = 3\Gamma(3) = 3.2.1,$$

Moreover, in general

$\Gamma(n+1) = n!,\ n$ a nonnegative integer.

This, of course, is what we wished to show. For future reference, we also note that

$$\begin{aligned}\Gamma(p+1)\left[(p+1)(p+2)\ldots(p+k)\right] &= \Gamma(p+2)[(p+2)..(p+k)]\\ &\vdots\\ &= \Gamma(p+k+1)\end{aligned} \tag{2.8}$$

Next, we observe that when Equation (1.40) is rewritten as

$$\Gamma(p) = \frac{\Gamma(p+1)}{p}, \tag{2.9}$$

It can be used to assign a value to $\Gamma(p)$ for nonintegral negative values of p. Indeed, Equation (2.9) can be used to define $\Gamma(p)$ for $-1 < p < 0$, sine $\Gamma(p+1)$ is already defined in that interval. Then continuing in the same fashion, $\Gamma(p)$ can be successively defined for $12 < p < -1, 13 < p < -2,$ and so forth.

Finally, it is easy to show that $\Gamma(p)$ is unbounded in the neighborhood of 0 and in neighborhood of every negative integer. More precisely,

$$\lim_{p\to 0^+}\Gamma(p) = +\infty \lim_{p\to 0^-}\Gamma(p) = -\infty,$$

With similar results about $-1, -2, \ldots$ A graph of the gamma function appears in Figure 2.

We Oliver and Maximau[15] now return to the discussion of Bessel's equation, and to Equation (2.2) when p is not an integer. This time we set.

$$a_0 = \frac{1}{2^p\Gamma(p+1)},$$

Moreover, use Equation (2.9) to obtain

$$J_{-p}x = \sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)\Gamma(k-p+1)}\left(\frac{x}{2}\right)^{2k-p} \tag{2.10}$$

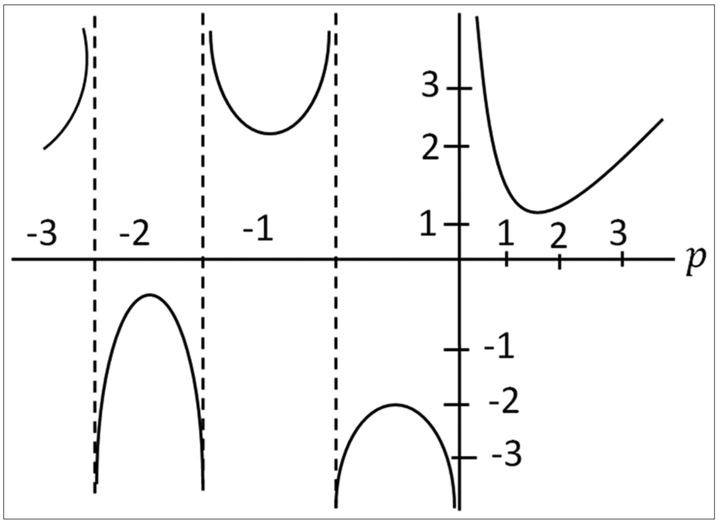

Figure 2: Isometric plotting of Bessel function of order p

Which is known as the Bessel function of order p of the first kind? It is defined for all x and reduces to our earlier formula for J_n when p is a non-negative integer n. To continue, we seek a second solution of Bessel's equation of order p that is linearly independent of J_p. The case where p in not an integer is easy. We simply replace p by $-p$ in the series for J_p to obtain.

$$J_{-p}(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)\Gamma(\mathrm{k}-\mathrm{p}+1)}\left(\frac{x}{2}\right)^{2k-p} \tag{2.11}$$

It is not difficult to verify that J_{-p} is a solution of Bessel's equation of order p and that it is linearly independent of J_p (See Exercise 1). Thus, when p is not an integer the general solution of Bessel's equation can be written in the form

$$y=c_1J_p+c_2J_{-p} \tag{2.12}$$

Where c_1 and c_2 are arbitrary constants? Note, however, that because of the term involving x^{-p}in (2.11), J_{-p} is unbounded near the origin and is undefined at $x=0$. Thus, Equation (2.12) is a solution of Bessel's equation only when $x>0$.

Bessel's function of the second kind

It is tempting to try to modify the dentition of J_{-p} so that when p is an integer n it will yield a solution of Bessel's equation that is linearly independent of J_n the most obvious way to try to do is to agree that.

$$J_{-n}(x)=\lim_{p\to n}J_{-p}(x) \tag{2.13}$$

It is easy to show that this limit exists but Pontryagin[16] unfortunately it turns our that

$$J_{-n}(x)=(-1)^nJ_n(x) \tag{2.14}$$

Thus J_{-n} is linearly dependent of J_n and our attempt has failed. At this point, we could of course return to first principles and obtain the required solution using the method of undetermined coefficients as described under the exceptional case of theorem 1.4. However, the corresponding computation is rather complicated, and what is worse, they result in function that is not particularly well suited for computation. As a result, we need to find a different approach to this problem. Surprisingly enough, a simple variant of the maneuver that just failed with J_{-p} will produce the solution we seek. In outline, the argument goes like. If p is not an integer, then the function.

$$Y_p(x)=\frac{J_p(x)\cos pn-J_{-p}(x)}{\sin p\pi} \tag{2.15}$$

Is defined for all $x > 0$ and being a linear combination of J_p and J_{-p} is a solution of Bessel's equation of order p. The important facts about Y_p are that it has a limit as $p \rightarrow n,$ and that in the limit it is still a solution of Bessel's equation. Thus, the function.

$$Y_n(x) = \lim_{p \rightarrow n} Y_n\ (x) \quad n = 0, 1, 2, \tag{2.16}$$

Is a solution of Bessel's equation of order n? What is more it is linearly independent of J_n? The function Y_n is known as the Bessel function of the second kind of order n.

The task of producing a series expansion for J_n is complicated and involves results that would take us too far afield to develop here. The computation begins with the observation that Equation (2.15) assumes the indeterminate form $\frac{0}{0}$ when $p = n,$ and hence that the limit in Equation (2.16) can be evaluated by ("Hopital") rule. Thus,

$$Y_n(x) = \frac{1}{\pi}\left[\frac{\partial}{\partial p} J_p(x) - (-1)^n \frac{\partial}{\partial p} J_{-p}(x)\right]_{p=n} \tag{2.17}$$

If the series for J_p and J_{-p} is substituted in this expression, differentiated, and the reindexed with n in place of p it can be shown Rabinstein[17] that.

$$Y_n(x) = \frac{2}{\pi} J_0(x)\left(In\frac{x}{2} + 7\right) - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n}$$

$$-\frac{1}{\pi(n!)}\left(2 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{\pi}\right)\left(\frac{x}{2}\right)^n$$

$$-\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k!(n+k)!}\left(\sum_{j=1}^{k}\frac{1}{j} + \sum_{J=1}^{k=n}\frac{1}{k}\right)\left(\frac{x}{2}\right)^{2k+n},$$

Where 7 = 0.57721566 … $= \lim_{n\rightarrow\infty}\left(1 + \frac{1}{2} + \ldots + 1/n - \text{In}\, n\right)$ is Euler's constant. In particular. When $n = 0$.

$$Y_0(x) = \frac{2}{\pi} J_0(x)\left(In\frac{x}{2} + 7\right) - \frac{2}{\pi}\sum_{k=0}^{n-1}\frac{(-1)^k}{(k!)^2}\left(1 + \frac{1}{2} + \ldots + \frac{1}{k}\right)\left(\frac{x}{2}\right)^{2k}$$

Graphs of Y_0 and Y_1 are shown in Figure 3.

Properties of bessel function

There is an almost endless list of formulas and identities that involve Bessel function. In this section, we shall establish a few of them.

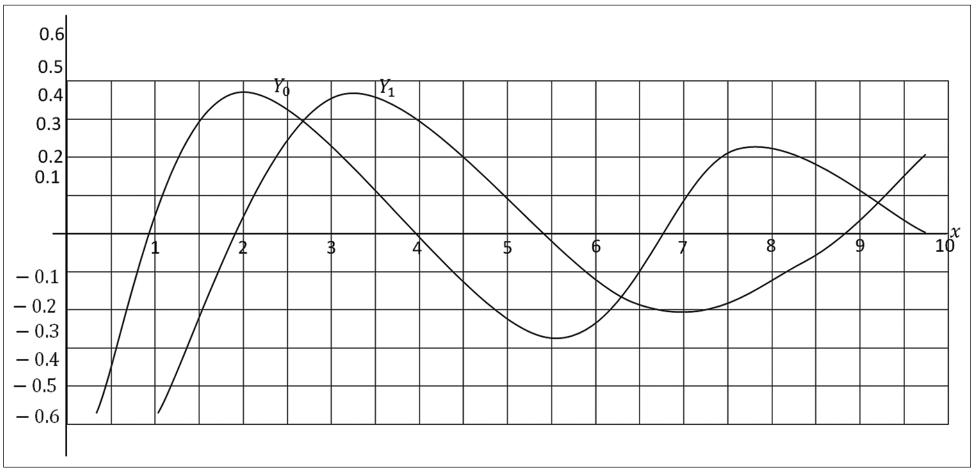

Figure 3: Struve's functions $H_o(z)$ and $H_1(z)$

Recurrence relations

The Rabinstein[17] – various recurrence relation that involves Bessel function follows from the differentiation formulas

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x) \tag{2.18}$$

Moreover,

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = x^{-p} J_{p-1}(x) \tag{2.19}$$

To prove Equation (1.52), we multiply the series expansion of $J_p \, by \, x^p$ and differentiate:

$$\frac{d}{dx}\left[x^p J_p(x)\right] = \frac{d}{dx}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)\Gamma(k+p+1)}\left(\frac{x}{2}\right)^{2k-2p}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k\ (k+p)}{\Gamma(k-1)\Gamma(k+p+1)}\left(\frac{x}{2}\right)^{2k-2p-1}$$

$$= x^p \sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)\Gamma(k+p+1)}\left(\frac{x}{2}\right)^{2k-2p-1}$$

$$= x^p J_{p-1}(x)$$

The proof of Equation (2.18) is similar.

When the derivatives appearing in Equations (2.18) and (2.19) are expanded, and the results are simplified we obtain

$$xJ'_p + pJ_p = xJ_{p-1} \tag{2.20}$$

and

$$xJ'_p - pJ_p = -xJ_{p+1} \tag{2.21}$$

The basic recurrence relations for the Bessel function follow at once from these results. By subtracting Equation (2.21) from (2.20), we obtain

$$2pJ'_p = xJ_{p-1} - xJ_{p+1}$$

By adding Equation (2.20) to (2.21), we obtain

$$2J'_p + J_{p-1} - J_{p+1}$$

Hence,

$$J_{p+1} = \frac{2p}{x} J_p - xJ_{p-1} \tag{2.22}$$

and

$$J'_p = \frac{1}{2}(J_{p-1} - J_{p+1} \tag{2.23}$$

Remake

Formulas Equation (2.19) through (2.22) also hold for the function Y_p . Bessel function of half-integral order. When $p = \frac{1}{2}$ Equation (2.15) becomes

$$J_{1/2}(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)\Gamma(\frac{3}{2}+k)}\left(\frac{x}{2}\right)^{2k+1/2}$$

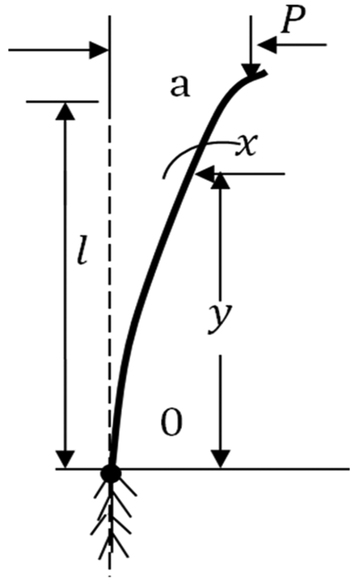

Figure 4: Struts of variable moment of inertia

$$=\sqrt{\frac{x}{2}}x^{p}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{2^{2\mathrm{k}}\Gamma(k)\Gamma\left(\frac{3}{2}+k\right)}x^{2k}.$$

However,

$$\Gamma\left(\frac{3}{2}+k\right)=\Gamma\left(\frac{3}{2}\right)\left[\frac{3}{2}.\frac{5}{2}\ldots\frac{2k+1}{2}\right]$$

$$=\Gamma\left(\frac{3}{2}\right)\left[\frac{3.5\ldots(2k+1)}{2^{k}}\right]$$

$$=\frac{\sqrt{\pi}}{2}\left[\frac{3\ .5\ldots(2k+1)}{2^{k}}\right],$$

Where the last step follows for the result in Exercise 2 of the preceding section. Hence,

$$J_{1/2}(x)=\sqrt{\frac{x}{2}}.\ \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{2^{\mathrm{k}}\,\mathrm{k}!3\ .5\ldots(2k+1)}x^{2k}$$

$$=\sqrt{\frac{2x}{\pi}}\sum_{k=0}^{\infty}(-1)^{k}\frac{x^{2k}}{(2k+1)}=\sqrt{\frac{2}{\pi x}}\sum_{k=0}^{\infty}(-1)^{k}\frac{x^{2k+1}}{(2k+1)!},$$

Moreover, we have proved that

$$J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x \tag{2.24}$$

A similar argument[17] reveals that

$$J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x, \tag{2.25}$$

Moreover, it now follows from Equation (2.22) that ever Bessel function of the first kind of order $n+\frac{1}{2}n$ an integer, can be written as a finite sum of terms involving powers of x and the sine or cosine. For instance,

$$J_{3/2}(x) = \frac{3}{x} J_{1/2}(x) - J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} \frac{3\cos x}{x} - \sin x \right).$$

It can be shown that the function of half-integral order is the only Bessel function that can be expressed in closed form in terms of elementary function. Problems of type considered in this work are an application to strut of variable moment of inertia, as illustrated below.

APPLICATION TO STRUT OF VARIABLE MOMENT OF INERTIA

This can be found in towers for supporting electrical transmission line, crane jibs, aircraft spars, locomotive connecting, and coupling rods.

To derive the Bessel's equation of struts of variable moment of inertia, if we take l_0 as the moment of inertia at a point 0 about an this perpendicular to the plane of bending, at (x, y) to be

$$l = l_0 e^{-ky/l} \tag{3.1}$$

Where k is a constant?

The bending moment at (x, y) is

$$m = p(a - x) \tag{3.2}$$

Moreover, the differential equation of flexure is

$$EI = \frac{d^2x}{dy^2} = M \tag{3.3}$$

E is the modulus of elasticity. Substituting from Equations (3.1) and (3.2) into (3.3), yields

$$\frac{d^2x}{dy^2} + \frac{e^{ky/l}}{El_0} p(x - a) = 0 \tag{3.4}$$

With $w = (x - a) 2b = k / l$ and $n^2 = p / El_0$ Equation (3.4) becomes

$$\frac{d^2w}{dy^2} + n^2 e^{2by} w = 0 \tag{3.5}$$

Writing $V = ne^{by}$ then Equation (3.5) is transformed to the Bessel equation

$$\frac{d^2w}{dv^2} + \frac{1}{v}\frac{dw}{dv} + \frac{w}{b^2} = 0 \tag{3.6}$$

solving the Bessel's equation above

Solution

$$\frac{d^2w}{dv^2} + \frac{1}{v}\frac{dw}{dv} + \frac{w}{b^2} = 0$$

Recall the standard Bessel's equation is

$$x^2 y^{11} + xy^1 + (x^2 - m^2) y = 0$$

Let $y^{11} = \frac{d^2w}{dv^2}, y^1 = \frac{dw}{dv}, y = w$ and $x = \frac{v}{b}$

$$\therefore y^{11} + \frac{1}{v} y^1 + \frac{1}{b^2} y = 0 \tag{3.7}$$

$$y^{11} + \frac{1}{v} y^1 + \left[\frac{1}{b^2} - 0y \right] = 0$$

$m = 0$ i.e. The Bessel function or order m

$$\therefore y^{11} + \frac{1}{v} y^{1} + \left[\frac{1}{b^2} - 0y \right] = 0$$

putting $y = \sum_{n=1}^{\infty} a_n x^{n+r}$ (3.8)

where $a_0 = 1$ and applying Frobenius method

$$y^{1} = \sum_{n=n}^{\infty} (n+r) a_n x^{n+r}$$

$$y^{11} = \sum_{n=n}^{\infty} (n+r-1)(n+r) a_n x^{n+r-2}$$

Substitution into the equation, we have

$$\sum_{n=\cap}^{\infty} (n+r-1)(n+r) a_n x^{n+r-2} + \frac{1}{v} \sum_{n=\cap}^{\infty} (n+r) a_n x^{n+r-1} + \frac{1}{b^2} \sum_{n=\cap}^{\infty} (n+r) a_n x^{n+r} = 0$$

Simplify, we have

$$\sum_{n=\cap}^{\infty} (n+r-1)(n+r) a_n x^{n+r-2} + \frac{1}{v} \sum_{n=\cap}^{\infty} (n+r) a_n x^{n+r-1} + \frac{1}{b^2} \sum_{n=\cap}^{\infty} (n+r) a_n x^{n+r} = 0$$

$$\sum_{n=\cap}^{\infty} (n+r-1)(n+r) a_n x^{n+r-2} + \frac{1}{b^2} \sum_{n=\cap}^{\infty} a_{n-2} x^{n+r} = 0 \quad (3.9)$$

Equation to zero the coefficient of the lowest power of $x (n+r)^2 (n+r-1) = 0$

We have $r^2 = 0$ which has equal roots

i.e. $r_1 = r_2 = 0$.

Thus, we have also

$$(n+r)^2 (n+r-1) a_n (r) + \frac{1}{b^2} a_{n-2} (r) = 0$$

$$a_n (r) = \frac{-1}{b^2} \frac{a_{n-2}(r)}{(n+r)^2 (n+r) z}, n \geq 2 \quad (3.10)$$

To determine $y_1 (x)$, we set $r = 0$, then from Equation (3.3) to zero the coefficient of higher power of x we obtain

$$\frac{1}{b^2} a_{n-2} = 0$$

i.e. $a_1 = 0$ for $r = 0$

Thus, from equation (40) we see that

$$a_3 = a_5 = a_7 \ldots = a_{2n+1} = \ldots = 0$$

Further

$$a_n (0) = \frac{-1}{b^2} \frac{a_{n-2}(0)}{n^2 (n-1)}, n = 2, 4, 6, 8. \quad \ldots$$

On letting $n = 2m$

$$a_m (0) = \frac{-1}{b^2} \frac{a_{2m-2}(0)}{m^2 (2m-1)}, m = 1, 2, 3 \quad \ldots$$

Thus, $a_2\left(0\right)=\frac{-1}{b^2}\frac{a_0}{2^2}=\frac{-1}{b^2}\frac{a_0}{2^2}$

$a_4\left(0\right)=\frac{-1}{b^2}\frac{a_{4-2}(0)}{(22)^2 3}=\frac{-1}{b^2}\frac{a_2(0)}{4^2.3}=\frac{+1}{b^2}\frac{a_0}{2(2)^2.3.3}$

$$a_{2m}\left(0\right)=\frac{-1}{b^2}\frac{a_0}{2^{2m}(m!)^2(m-1)} \tag{3.11}$$

Hence,

$$Y_1\left(x\right)=\left[a_0\left\{|+\frac{(-1)^m x^{2m}}{\left(m!\right)^2(m-1)}\right\}\right]\frac{1}{b^2}x>0 \tag{3.12}$$

To get the second linearly independent solution, we compute $a_n^1\left(0\right)$

Note from the coefficient of x^r

$(r+1)^2 a_1\left(r\right)=0$

If follows that

$a_1\left(0\right)=0\, a_1^1\left(0\right)=0$

From recurrence relation Equation (2.10)

$a_3^1\left(0\right)=a_5^1=\ldots=a_{2+1}\left(0\right)=0 \quad \ldots=0$

We need only to complete $a_{2m}^1\left(0\right), M=1,2,3,\ldots$

$\therefore a_2\left(r\right)=\frac{-1}{b^2}\frac{a_{2m-2}\ (r)}{(2m+r)^2\ (m+r-1)}, m=1,2,3,$

$\Rightarrow \frac{-a_0}{b^2(2+r\ )^{2r}}=a_4\left(r\right)=\frac{-1}{b^2}\frac{a_2(r)}{\left(4+r\right)^r\ (2+r-1)}=\frac{+a_0(r)}{b^4\ (4+r)^2\ (2+r)^2\left(r+1\right)r}$

$\Rightarrow a_{2m}\left(r\right)=\frac{(-1)^m a_0}{b^{2m}\ \{(2m+r)^2(2m-r+1)^2}$

This computation of $a_{2m}\left(r\right)$ can be done most conveniently by noting that if

$f\left(x\right)=\left(x-\propto_1\right)^{\beta}\left(x-\propto_2\right)^{\beta 2}\quad \ldots\left(x-\propto_n\right)^{\beta n}$

$f^1\left(x\right)=\beta_1\left(x-\propto_1\right)\text{verify}\,\beta^{1.1}\beta^{2.1}\,\text{or}\beta^{i-1}\beta^{i-2}\left[\left(x-\propto_2\right)^{\beta 2}\quad \ldots\left(x-\propto_n\right)^{\beta n}\right]+\ldots$

$\beta_2\left(x-\propto_2\right)^{\beta 2-1}\left[\left(x-\propto_3\right)^{\beta 3}\ldots\left(x-\propto_n\right)^{\beta n}\right]+\ldots$

Hence, for x not equal to $\propto_1,\propto_2\ldots\propto_n$

$\frac{f^1\left(x\right)}{f\left(x\right)}=\frac{\beta_1}{x-\propto_1}+\frac{\beta_2}{x-\propto_2}+\ldots\frac{\beta_n}{\propto_n}$

Moreover, letting r equal zero we obtain

$$a_{2m}^{1}(0)=\left(\frac{-1}{2m(m-1)}+\frac{1}{2(m-1)(m-1)}+\ldots+\frac{1}{2(m-1)}\right)\frac{a_{2m}(0)}{b^{2m}}$$

Substitution for $a_{2m}^{(0)}$ from Equation 2.6

$$H_m=\left(\frac{1}{m(m-1)}+\frac{1}{(m-1)(m-1)}+\ldots+\frac{1}{2(m-1)}+1\right)\frac{1}{b^m}$$

We finally obtain

$$a_{2m}^{1}(0)=\frac{-H_m}{b^2}\frac{(-1)^m a_0}{2^{2m}(m!)^2(m-1)},m=1,2,3, \qquad (3.13)$$

The second solution of the Bessel's equation of order zero is obtained by setting $a_0=1$ and substituting $r_1(x)$ and $a_{2m}(0)=b_{2m}(0)$ in

$$Y_2(x)=Y_1(x)l_n|x|+|x|^{r_2}\sum_{n=\cap}^{\infty}b_n(r)x^n$$

$$Y_2(x)=J_0(x)|nx\sum_{n=\cap}^{\infty}\frac{(-1)n+1}{22n\ (n!)^2}x\,2n,x>0$$

Recall the general solution is written as

$$Y=a_1J_0(x)+a_2Y_0(x)$$
$$\therefore Y=a_1J_0(v/x)+a_2Y_0(v/b)$$

Which is the solution condition?

$$W=(x-a)=AJ_1(v/b)+BY_0(v/b) \qquad (3.14)$$

And hence, the general solution

$$\frac{dx}{dy}=(x-a)=-v[AJ_1\ (v/b)+BY_1\ (v/b)] \qquad (3.15)$$

Table 1: Numerical values pertaining to (3.15)

I_l/I_0	$k=\log_e(I_0/I_l)$	$\varphi=e^{\frac{1}{2}k}$	$\theta_c=\frac{2l}{k}\left(\frac{P_c}{EI_0}\right)^{\frac{1}{2}}$	φ^{θ_c}	$\propto$
0.025	3.6889	6.3245	0.3973	2.5127	0.5370
0.05	2.9957	4.4721	0.5848	2.6153	0.7674
0.10	2.3026	3.1623	0.894φ7	2.8293	1.0610
0.20	1.6904	2.2361	1.4800	3.3094	1.4183
0.30	1.2040	1.8257	1.4800	3.3094	1.6557
0.40	0.91630	1.5811	2.9574	4.6757	1.8359
0.50	0.69315	1.4142	4.0618	5.7442	1.9817
0.60	0.51083	1.2910	5.6803	7.3333	2.1049
0.70	0.35668	1.1952	8.3401	9.9681	2.2123
0.80	0.22314	1.1180	13. 6137	15.2201	2.3070
0.90	0.10536	1.0541	29.3426	30.9300	2.3894
1.00	0.0000	1.0000			2.4674
					$\simeq\frac{1}{2}\pi^2$

Putting

$$x = a \text{ at } y = 1 \text{ and } \frac{dx}{dy} = 0 \text{ at } y = 0 \tag{3.16}$$

Inserting into the general solution

$$AJ_0\ (2/ne^{1/2k/k}) + BY_0\left(2\middle|ne^{1/2}\right) = 0 \tag{3.17}$$

$$AJ_1(2\ln lk) + BY_1(2\ln lk) = 0 \tag{3.18}$$

Equation the valves of B/A from Equations (3.17) and (3.18) lead to

$$J_0(\varphi\theta)Y_1(\theta) = Y_0(\varphi\theta)J_1(\theta) = 0 \tag{3.19}$$

Let $\varphi = e^{1/2\,k}$, $\theta = n/b = 2ln\ /k, n^2 = P/EI_0$

$$\therefore \theta_c = 2l_n c/k = (2l/k)(P_c/E1_0)^{1/2} \tag{3.20}$$

Moreover, the critical thrust is

$$p_c = \propto^2 \frac{EI_0}{|^2} \tag{3.21}$$

Where $\propto = \frac{k\theta_c\cdot}{2}$ for a uniform thrust $\propto = \frac{1}{2}\pi$

Numerical data

These are given in Table 1, and they cover a fairly wide range of $k, \propto$. The critical load for a strut symmetrical about the midpoint of its length, and having hinged ends, is obtained from Equation (3.6), by writing $l/2$ for l. They we get

$$P'_c = 4\propto^2 EI_0/l^2$$

Moreover, for a uniform strut $4\propto^2 = \pi^2$, which yields the Eulerian value of P'_c.

Numerical values pertaining to Equation (3.4).

Degenerate case when k $\rightarrow 0$***s***

The argument of the Bessel function in Equation (3.4) now tend to infinity, so the Bessel's function using Equations (3.1) and (3.2) in the equation we get may be replaced by the dominant terms in their asymptotic expansion. Using Equation (3.1), Equation (3.2) we get

$$\cos\left(\varphi\theta = \frac{1}{4}\pi\right)\sin\left(\theta - \frac{3}{4}\pi\right) - \sin\left(\varphi\theta - \frac{1}{4}\pi\right)\cos\left(\theta - \frac{3}{4}\pi\right) = 0 \tag{3.22}$$

Or

$$\sin\left[\theta(\varphi - 1) + \frac{1}{2}\pi\right] = 0, \tag{3.23}$$

Moreover, the smallest positive root is given by

$$\theta_c(\varphi - 1) + \frac{1}{2}\pi = \pi \tag{3.24}$$

However,

$$(\varphi - 1) = \left(e^{\frac{1}{2}} - 1\right) = \frac{1}{2}k + \frac{\left(\frac{1}{2}k\right)^2}{2!} + \ldots \tag{3.25}$$

Hence, with θ_c from Equation (3.20) the left-hand side of Equation (3.21) may be written

$$(2l/k)(P_c / EI_0)^{\frac{1}{2}}\left\{\frac{1}{2}k + \frac{(\frac{1}{2}k)^2}{2!} + \ldots\right\} + \frac{1}{2}\pi \to l(P_c / ELI_0)^{\frac{1}{2}} + \frac{1}{2}\pi, \tag{3.26}$$

As $k \to 0$. Hence, equating the right-hand sides of Equations (3.24) and (3.26) we obtain

$$P_c = \frac{1}{4}\frac{\pi^2 EL_0}{l^2}, \tag{3.27}$$

The well-known Eulerian buckling load.

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