ISSN 2581-3463

AJMS
Asian Journal of Mathematical Sciences

RESEARCH ARTICLE

On Certain Special Vector Fields in a Finsler Space III

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Received: 15-07-2019; Revised: 15-09-2019; Accepted: 19-10-2019

ABSTRACT

In an earlier paper in 2017, Rastogi and Bajpai[1] defined and studied a special vector field of the first kind in a Finsler space as follows:

Definition 1: A vector field $X^i(x)$, in a Finsler space, is said to be a special vector field of the first kind, if (i) $X^i_{/j} = - \delta^i_j$ and (ii) $X^i h_{ij} = \Theta_j$, where Θ_j is a non-zero vector field in the given Finsler space.

In 2019, some more special vector fields in a Finsler space of two and three dimensions have been defined and studied by the authors Dwivedi *et al*.[2] and Dwivedi *et al*.[3] In Dwivedi *et al*.[3], the authors defined and studied six kinds of special vector fields in a Finsler space of three dimensions and, respectively, called them special vector fields of the second, third, fourth, fifth, sixth, and seventh kind. In the present paper, we shall study some curvature properties of special vector fields of the first and seventh kind in a Finsler space of three dimensions.

Key words: Curvature tensors, Finsler space,vector fields

INTRODUCTION

Let F^3, be a three-dimensional Finsler space, with metric function L(x,y), metric tensor $g_{ij} = l_i l_j + m_i m_j + n_i n_j$ and angular metric tensor $h_{ij} = m_i m_j + n_i n_j$, where $l_i = \Delta_i L$ and $\Delta_i = \partial/\partial y^i$, while m_i and n_i are vectors orthogonal to each other Matsumoto.[4] The torsion tensor $A_{ijk} = L C_{ijk} = (L/2) \Delta_k g_{ij}$. The h- and v-covariant derivatives of a tensor field $T^i_j(x,y)$ are defined as Matsumoto:[4]

$$T^i_{j/k} = \partial_k T^i_j - N^m_k \Delta_m T^i_j + T^m_j F^i_{mk} - T^i_m F^m_{jk} \tag{1.1}$$

and

$$T^i_{j//k} = \Delta_k T^i_j + T^m_j C^i_{mk} - T^i_m C^m_{jk} \tag{1.2}$$

where, $\partial_k = \partial/\partial x^k$ and other terms have their usual meanings Matsumoto.[4]
Corresponding to h- and v-covariant derivatives, in F^3, we have:

$$l_{i/j} = 0, \; m_{i/j} = n_i h_j, \; n_{i/j} = - m_i h_j, \tag{1.3}$$

and

$$l_{i//j} = L^{-1} h_{ij}, \; m_{i//j} = L^{-1}(-l_i m_j + n_i v_j), \; n_{i//j} = - L^{-1}(l_i n_j + m_i v_j) \tag{1.4}$$

where, h_j and v_j are respectively h- and v-connection vectors in F^3. Furthermore,

$$C_{ijk} = C_{(1)} m_i m_j m_k - \sum_{(I,j,k)} \{C_{(2)} m_i m_j n_k - C_{(3)} m_i n_j n_k\} + C_{(2)} n_i n_j n_k \tag{1.5}$$

Corresponding to these covariant derivatives, we have following:

$$T^i_{j/k//h} - T^i_{j//h/k} = T^r_j P^i_{rkh} - T^i_r P^r_{jkh} - T^i_{j/r} C^r_{kh} - T^i_{j//r} P^r_{kh} \tag{1.6}$$

and

$$T^i_{j//k//h} - T^i_{j//h//k} = T^r_j S^i_{rkh} - T^i_r S^r_{jkh} - T^i_{j//r} S^r_{kh} \tag{1.7}$$

where, P^r_{jkh} and S^r_{jkh} are, respectively, the second and third curvature tensors of F^3, Rund.[5]

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PROPERTIES OF SPECIAL VECTOR FIELD OF THE FIRST KIND RELATED WITH THE SECOND CURVATURE TENSOR

In F^3, we assume

$$X^i(x) = A\, l^i + B\, m^i + D\, n^i, \tag{2.1}$$

where, A, B, and D are scalars satisfying $X^i\, l_i = A$, $X^i\, m_i = B$, and $X^i\, n_i = D$ such that for $X^i_{/j} = -\delta^i_j$, we get

$$A_{/j} = -l_j,\ B_{/j} = D\, h_j - m_j,\ D_j = -(B\, h_j + n_j) \tag{2.2}$$

and

$$A_{//j} = L^{-1}(B\, m_j + D\, n_j),$$
$$B_{//j} = (C_{(1)}\, B - C_{(2)}\, D - L^{-1}\, A)\, m_j + (C_{(3)}\, D - C_{(2)}\, B)\, n_j + L^{-1}\, D\, v_j$$
$$D_{//j} = (C_{(3)}\, D - C_{(2)}\, B)\, m_j + (C_{(3)}\, B + C_{(2)}\, D - L^{-1}\, A)\, n_j - L^{-1}\, B\, v_j \tag{2.3}$$

From definition 1., we can obtain $\Theta_j = B\, m_j + D\, n_j$, $\Theta_{j/k} = -h_{jk}$. Furthermore, we get

$$\Theta_{j//r} = (C_{(1)}\, B - C_{(2)}\, D)\, m_j\, m_r + (C_{(3)}\, B + C_{(2)}\, D)\, n_j\, n_r$$
$$+ (C_{(3)}\, D - C_{(2)}\, B)\,(m_j\, n_r + m_r\, n_j) - L^{-1}(A\, h_{jr} + l_j\, \Theta_r). \tag{2.4}$$

These equations help us to give

$$\Theta_{j/k//r} = L^{-1}(l_j\, h_{kr} + l_k\, h_{jr}) \tag{2.5}$$

and

$$\Theta_{j//r/k} = \{(C_{(1)/k} + 3\, C_{(3)}\, h_k)\, B - (C_{(2)/k} - (C_{(1)} - 2\, C_{(3)})\, h_k)$$
$$- C_{(1)}\, m_k + C_{(2)}\, n_k\}\, m_j\, m_r + \{(C_{(3)/k} - 3\, C_{(2)}\, h_k)B$$
$$+ (C_{(2)/k} + 3\, C_{(3)}\, h_k) - C_{(3)}\, m_k - C_{(2)}\, n_k\}\, n_j\, n_r + \{(C_{(3)/k} - 3\, C_{(2)}\, h_k)D$$
$$- (C_{(2)/k} + (2\, C_{(3)} - C_{(1)})h_k)B + C_{(2)}\, m_k - C_{(3)}\, n_k\}\}\,(m_j\, n_r + m_r\, n_j)$$
$$+ L^{-1}\,(l_j\, h_{kr} + l_k\, h_{jr}) \tag{2.6}$$

Using Equations (1.6), (2.5), and (2.6) on simplification, we obtain

$$\Theta_t\, P^t_{jkr} + \Theta_{j//t}\, P^t_{kr} - \{(C_{(1)/k} + 3\, C_{(3)}\, h_k)\, B$$
$$- (C_{(2)/k} - (C_{(1)} - 2\, C_{(3)})\, h_k)\, D\}\, m_j\, m_r - \{(C_{(3)/k} - 3\, C_{(2)}\, h_k)B$$
$$+ (C_{(2)/k} + 3\, C_{(3)}\, h_k)\, D\}\, n_j\, n_r - \{(C_{(3)/k} - 3\, C_{(2)}\, h_k)D$$
$$- (C_{(2)/k} + (2\, C_{(3)} - C_{(1)})h_k)B\}\,(m_j\, n_r + m_r\, n_j) = 0. \tag{2.7}$$

Hence:

Theorem 2.1

In a three-dimensional Finsler space F^3, for a special vector field of the first kind, the second curvature tensor P^t_{jkr} satisfies Equation (2.7).

PROPERTIES OF SPECIAL VECTOR FIELDS OF THE FIRST KIND RELATED WITH THE THIRD CURVATURE TENSOR

From Equation (2.4), we can get on simplification

$$\Theta_{j//r//k} = B\{C_{(1)//k}\, m_j\, m_r + C_{(3)//k}\, n_j\, n_r - C_{(2)//k}(m_j\, n_r + m_r\, n_j)\}$$
$$+ D\{C_{(2)//k}\, n_j\, n_r - C_{(2)//k}\, m_j\, m_r + C_{(3)//k}\,(m_j\, n_r + m_r\, n_j)\}$$
$$+ B_{//k}\{C_{(1)}\, m_j\, m_r + C_{(3)}\, n_j\, n_r - C_{(2)}(m_j\, n_r + m_r\, n_j)\}$$
$$+ D_{//k}\{C_{(2)}\, n_j\, n_r - C_{(2)}\, m_j\, m_r + C_{(3)}\,(m_j\, n_r + m_r\, n_j)\}$$
$$+ m_{j//k}\{(C_{(1)}\, B - C_{(2)}\, D - L^{-1}\, A)\, m_r + (C_{(3)}\, D - C_{(2)}\, B)\, n_r\}$$
$$+ n_{j//k}\{(C_{(3)}\, B + C_{(2)}\, D - L^{-1}\, A)\, n_r + (C_{(3)}\, D - C_{(2)}\, B)\, m_r\}$$
$$+ m_{r//k}\{(C_{(1)}\, B - C_{(2)}\, D - L^{-1}\, A)\, m_j + (C_{(3)}\, D - C_{(2)}\, B)\, n_j\}$$
$$+ n_{r//k}\{(C_{(3)}\, B + C_{(2)}\, D - L^{-1}\, A)\, n_j + (C_{(3)}\, D - C_{(2)}\, B)\, m_j\}$$
$$+ L^{-2}\{l_k(A\, h_{jr} + l_j\, \Theta_r) - L\, A_{//k}\, h_{jr} - h_{jk}\, \Theta_r - L\, l_j\, \Theta_{r//k}\} \tag{3.1}$$

Using Equations (1.7) and (3.1), after some lengthy calculation, we can obtain

$$\varsigma_{(k,r)}\,[B\{(C_{(1)//r} + 2\, L^{-1}\, C_{(2)}\, v_r)\, m_j\, m_k - (C_{(2)//r} + 2\, L^{-1}\, C_{(3)}\, v_r)\, m_j\, n_k$$
$$+ L^{-1}C_{(1)}\, m_j\,(l_r\, m_k + v_r\, n_k) - L^{-1}C_{(2)}\, m_j\,(l_r\, n_k + m_r\, v_k) + (C_{(3)//r}$$
$$- 3\, L^{-1}\, C_{(2)}\, v_r)\, n_j\, n_k - (C_{(2)//r} + (C_{(1)}\, C_{(3)} - 2\, C_{(2)}{}^2 - C_{(3)}{}^2)n_r)\, n_j\, m_k$$
$$+ L^{-1}\, C_{(3)}\,(l_r\, n_k + m_r\, v_k)\, n_j - L^{-1}((C_{(3)} - C_{(1)})\, v_r + C_{(2)}\, l_r)n_j\, m_k$$

$$- L^{-2}n_j n_k m_r + L^{-1}l_j l_r m_k\} + D\{(C_{(3)//r} - 3 L^{-1}C_{(2)} v_r + L^{-1} C_{(3)} l_r)m_j n_k$$
$$- (C_{(2)//r} + (C_{(1)} C_{(3)} + C_{(3)}{}^2) + L^{-2})n_r - L^{-1}((2 C_{(3)} - C_{(1)}) v_r + C_{(2)} l_r) m_j m_k$$
$$+ (C_{(2)//r} - 2 L^{-1}C_{(2)} v_r + L^{-1} C_{(3)} l_r) n_j m_k + (C_{(3)//r} + L^{-1} (3C_{(3)} v_r$$
$$+ C_{(2)} l_r))n_j n_k + L^{-1} (C_{(2)} n_j m_r v_k - L^{-1}l_j l_k n_r)\} - L^{-2}A\{(l_r n_k + m_r v_k) m_j$$
$$- (l_r n_k - v_r m_k) n_j\}] + \Theta_p S^p_{jkr} + \Theta_{j//p} S^p_{kr} = 0. \quad (3.2)$$

Hence:

Theorem 3.1

In a three-dimensional Finsler space F^3, for a special vector field of the first kind, the third curvature tensor S^t_{jkr} satisfies Equation (3.2).

PROPERTIES OF SPECIAL VECTOR FIELD OF THE SEVENTH KIND

The special vector field of the seventh kind is defined as follows:[3]

Definition 2

A vector field $X^i(x)$, satisfying i) $X^i_{/j} = - \delta^i_j$ and $X^i Y_{ij} = Y_i$, where Y_i is a non-zero vector field in F^3 and $Y_{ij} = m_i n_j - m_j n_i$ is a tensor field, is called special vector field of the seventh kind.
From this definition, we can observe that

$$Y_j = B n_j - D m_j, Y_{j/k} = Y_{jk} \quad (4.1)$$

and

$$Y_{j//k} = (C_{(1)} B - C_{(2)} D) m_k n_j - (C_{(3)} B + C_{(2)} D) m_j n_k + L^{-1}\{A (m_j n_k - m_k n_j)$$
$$- l_j Y_k\} + (C_{(3)} D - C_{(2)} B)(n_j n_k - m_j m_k) \quad (4.2)$$

From Equation (4.1), we can obtain

$$Y_{j/k//r} = L^{-1}(l_j Y_{kr} + l_k Y_{jr}) \quad (4.3)$$

while from Equation (4.2), we get

$$Y_{j//r/k} = \{C_{(1)/k} B - C_{(2)/k} D + C_{(1)}(D h_k - m_k) + C_{(2)}(B h_k + n_k)\}m_r n_j$$
$$- ((C_{(1)} B - C_{(2)} D)(n_j n_r - m_j m_r) h_k + \{C_{(3)/k} B + C_{(2)/k} D$$
$$+ C_{(3)}(D h_k - m_k) - C_{(20)} (B h_k + n_k)\}m_j n_r + (C_{(3)} B + C_{(2)} D).$$
$$(n_j n_r - m_j m_r) h_k + L^{-1} \{l_k(m_j n_r - m_r n_j) + l_j Y_{rk}\} - \{C_{(3)/k} D$$
$$- C_{(2)/k} B - C_{(3)} (B h_k + n_k) - C_{(2)} (D h_k - m_k)\}(n_j n_r - m_j m_r)$$
$$+ 2(C_{(3)} D - C_{(2)} B)(m_j n_r + m_r n_j) h_k \quad (4.4)$$

Equations (4.3) and (4.4) with the help of Equation (1.6) lead to

$$B[(C_{(3)/k} - 3 C_{(2)} h_k) m_j n_r - (C_{(1)/k} + 3 C_{(2)} h_k) m_r n_j$$
$$+ \{C_{(2)/k} + (2 C_{(3)} - C_{(1)}) h_k\}(n_j n_r - m_j m_r)]$$
$$+ D[\{C_{(2)/k} + (2 C_{(3)} - C_{(1)}) h_k\} m_r n_j + (C_{(2)/k} + 3 C_{(3)} h_k) m_j n_r$$
$$-(C_{(3)/k} - 3 C_{(2)} h_k)(n_j n_r - m_j mr)] - m_k\{C_{(2)} (n_j n_r - m_j m_r)$$
$$- C_{(1)} m_r n_j + C_{(3)} m_j n_r\} + n_k \{C_{(3)}(n_j n_r - m_j m_r)$$
$$- C_{(2)}(m_r n_j + m_j n_r)\} + Y_t P^t_{jkr} + Y_{j//t} P^t_{kr} = 0. \quad (4.5)$$

Hence:

Theorem 4.1

In a three-dimensional Finsler space F^3, for a special vector field of the seventh kind, the second curvature tensor satisfies Equation (4.5).
From Equation (4.2), we can get

$$Y_{j//k//r} = m_j[\{C_{(2)//r} B - C_{(3)//r} D + C_{(2)} B_{//r} - C_{(3)} D_{//r})m_k + (C_{(2)} B - C_{(3)} D).$$
$$L^{-1}(-l_k m_r + n_k v_r)\} - \{C_{(3)//r} B + C_{(2)//r} D - L^{-1}(B m_r + D n_r) + L^{-2} A l_r$$
$$+ C_{(3)} B_{//r} + C_{(2)} D_{//r})\} n_k + (C_{(3)} B + C_{(2)}D - L^{-1}A) L^{-1}(l_k n_r + m_k v_r)]$$

$$
\begin{aligned}
&+ n_j[\{(C_{(1)//r} B - C_{(2)//r} D + C_{(1)} B_{//r} - C_{(2)} D_{//r} + L^{-2} A\, l_r \\
&- L^{-2}(B\, m_r + D\, n_r)\}\, m_k + L^{-1}(-l_k m_r + n_k v_r)(C_{(1)} B - C_{(2)} D - L^{-1}A) \\
&+(C_{(3)//r} D - C_{(2)//r} B + C_{(3)} D_{//r} - C_{(2)} B_{//r})n_k + L^{-1}(C_{(3)} D - C_{(2)} B). \\
&(l_k n_r + m_k v_r)] + L^{-1}(-l_j m_r + n_j v_r)\{(C_{(2)} B - C_{(3)} D)\, m_k - (C_{(3)} B \\
&+ C_{(2)} D - L^{-1}A)\, n_k\} - L^{-1}(l_j n_r + m_j v_r)\{(C_{(1)} B - C_{(2)} D - L^{-1}A)m_k \\
&+ (C_{(3)} D - C_{(2)} B)\, n_k\} + L^{-2}\, Y_k(l_j l_r - h_{jr}) - L^{-1}\, l_j\, Y_{kr},
\end{aligned}
\tag{4.6}
$$

which by virtue of Equation (1.7) after some lengthy calculation leads to

$$
\begin{aligned}
&Y_t\, S^t_{jkr} + \mathfrak{S}_{(k,r)} [m_j m_k\{B(C_{(2)//r} - (C_{(2)}^2 - C_{(3)}^2)\, n_r - L^{-1}(C_{(1)} + C_{(2)})\, v_r) \\
&- D(C_{(3)//r} - L^{-1}C_{(2)} v_r) - L^{-1}A\, C_{(3)} n_r\} - n_j n_k\{B(C_{(2)//r} + C_{(2)} (C_{(1)} + C_{(3)})m_r \\
&+ 2 L^{-1}C_{(3)} v_r) + D(C_{(3)//r} - (C_{(2)}^2 + C_{(3)}^2)\, m_r - L^{-1} C_{(2)} v_r) + L^{-1}A\, C_{(2)} m_r\} \\
&+ m_j n_k\{B(C_{(2)}^2 - C_{(1)} C_{(3)})m_r - C_{(3)//r} + L^{-1}m_r + L^{-1}(C_{(1)} + 3\, C_{(2)})\, v_r) \\
&+ D(C_{(2)//r} + 2\, C_{(2)} C_{(3)}\, m_r - L^{-1}m_r - L^{-1}(C_{(2)} - 3\, C_{(3)})\, v_r) + L^{-1} A\, C_{(3)} m_r\} \\
&- m_k n_j\, \{B(C_{(1)//r} + C_{(2)}(C_{(1)} - C_{(3)})\, n_r) + D(C_{(2)//r} - (C_{(2)}^2 + C_{(1)} C_{(3)})\, n_r \\
&- L^{-1}C_{(1)} v_r) - L^{-1}A\, C_{(2)} n_r\} - L^{-1}l_k\{C_{(3)} h_{jr} + C_{(2)}(m_j n_r + m_r n_j)\}] = 0.
\end{aligned}
\tag{4.7}
$$

Hence:

Theorem 4.2

In a three-dimensional Finsler space, for a special vector field of the seventh kind, the third curvature tensor S^t_{jkr} satisfies Equation (4.7).

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