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RESEARCH ARTICLE

An Extension of Calderón Transfer Principle and its Application to Ergodic Maximal Function

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ABSTRACT

We first prove that the well-known transfer principle of Calderón can be extended to the vector-valued setting, and then, we apply this extension to vector-valued inequalities for the Hardy–Littlewood maximal function to prove the vector-valued strong type L^p norm inequalities for $1 < p < \alpha$ and the vector-valued weak type (1,1) inequality for ergodic maximal function.

Key words: Transfer principle, translation invariant operator, vector-valued inequality

INTRODUCTİON

Vector-valued inequalities have long been studied in harmonic analysis, and unfortunately, there is not so many results in this direction in ergodic theory. The well-known transfer principle of Calderón[1] is a great tool to prove certain type of inequalities in ergodic theory. Using this transfer principle, one can use certain type of weak type or strong type inequalities in harmonic analysis to prove analog weak type or strong type inequalities in ergodic theory under certain assumptions. It is, therefore, quite natural to look for a Calderón type transfer principle to be able to prove vector-valued inequalities in ergodic theory. Unfortunately, long time past after the invention of the transfer principle of Calderón[1] used to prove classical weak type or strong type inequalities for some well known operators in ergodic theory by using the corresponding inequalities for some operators in harmonic analysis but a transfer principle to prove vector-valued weak type or strong type inequalities for those operators in ergodic theory satisfying the conditions of Calderón transfer principle has not been developed even though vector-valued inequalities are much more general than classical inequalities. The purpose of this work is to fill this gap by proving that the Calderón transfer principle can be extended to be able to prove vector-valued inequalities in ergodic theory using vector-valued inequalities in harmonic analysis. Even though our proof is an extension of Calderón's argument, the contribution of our result to analysis is remarkable when proving vector-valued inequalities in ergodic theory. After proving that, Calderón transfer principle can be extended to the vector-valued inequalities, we apply our result to a theorem of Fefferman and Stein[2] to prove the vector-valued strong L^p inequalities for $1 < p < \alpha$ and the vector-valued weak type (1,1) inequality for the ergodic maximal function. It is was also proved by the author in Demir[3] that the transfer principle of Calderón can be generalized to prove L^p norm inequalities in-between two operators in ergodic theory under certain assumptions.

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THE SETUP

We will assume that X is a measure space which is totally σ -finite and U^t is a one-parameter group of measure-preserving transformations of X. We will also assume that for every measureable function f on $g(x)=G(0,x)$, the function $f(U^t x)$ is measurable in the product of X with the real line. T will denote an operator defined on the space of locally integrable functions on the real line with the following properties: the values of T are continuous functions on the real line, T is sublinear and commutes with translations, and T is semilocal in the sense that there exists a positive number ε such that the support of Tf is always contained in an ε-neighborhood of the support of f. We will associate an operator $T^{\#}$ on the functions on X with such an operator T as follows: Given a function f_j on X let

$$F_j(t,x) = f_j(U^t x).$$

If f_j is the sum of two functions which are bounded and integrable, respectively; then, $F_j(t,x)$ is a locally integrable function of t for almost all x and, therefore,

$$G_j(t,x) = T(F_j(t,x))$$

is a well-defined continuous function of t for almost all x. Thus, $g_i(x)=G_j(0,x)$ has a meaning, and we define

$$T^{\#} f_j = g_j(x).$$

Let now (T_n) be a sequence of operators as above and define

$$Sf = \sup_n |T_n f|$$

and

$$S^{\#} f = \sup_n |T_n^{\#} f|.$$

Suppose that $f_= (f_1, f_2, f_3, \ldots)$ then we have $Tf_= (Tf_1, Tf_2, Tf_3, \ldots)$. Let now $1 \leq r < \alpha$ then

$$\|f(x)\|_{l^r} = \left(\sum_{j=1}^{\infty} \left| f_j(x) \right|^r \right)^{1/r}$$

and similarly

$$\|Tf(x)\|_{l^r} = \left(\sum_{j=1}^{\infty} \left| Tf_j(x) \right|^r \right)^{1/r}.$$

Also, note that

$$\|f(x)\|_{l^\infty} = \sup_j \left| f_j(x) \right|$$

and

$$\|Tf(x)\|_{l^\infty} = \sup_j \left| Tf_j(x) \right|.$$

THE RESULT

We can now state and prove our main result as follows:

Theorem 1

Let $1 \leq r \leq \alpha$ and $1 \leq p \leq \alpha$. Suppose that there exists a constant $C_1 > 0$ such that

$$\left(\int_{\mathbb{R}} \|Sf(t)\|_{l^r}^p \right)^{1/p} \leq C_1 \left(\int_{\mathbb{R}} \|f(t)\|_{l^r}^p \, dt \right)^{1/p}$$

Where $f_= (f_1, f_2, f_3, \ldots)$ is a sequence of functions on $\mathbb{R}$. Then, we have

$$\left(\int_X \|S^{\#} f(x)\|_{l^r}^p \, dx \right)^{1/p} \leq C_1 \left(\int_X \|f(x)\|_{l^r}^p \, dx \right)^{1/p}$$

where $f_= (f_1, f_2, f_3, \ldots)$ is a sequence of functions on X. Suppose that there exists a constant $C_2 > 0$ such that for all $\lambda > 0$

$$\left| \left\{ t \in \mathbb{R} : \|Sf(t)\|_{l^r} > \lambda \right\} \right| \leq \frac{C_2^p}{\lambda^p} \int_{\mathbb{R}} \|f(t)\|_{l^r}^p dt$$

where $f_= (f_1, f_2, f_3, \ldots)$ is a sequence of functions on $\mathbb{R}$. Then for all $\lambda > 0$ we have

$$\left| \left\{ x \in X : \|S^{\#} f(x)\|_{l^r} > \lambda \right\} \right| \leq \frac{C_2^p}{\lambda^p} \int_X \|f(x)\|_{l^r}^p dx$$

Where $f_= (f_1, f_2, f_3, \ldots)$ is a sequence of functions on X.

Proof

We apply the argument of Calderón[1] to the vector-valued setting to prove our theorem.
Without loss of generality, we may assume that the sequence (T_n) is finite, for if the theorem is established in this case, the general case follows by a passage to limit. Under this assumption, the operators S have the same properties as the operators T above. Let

$$F(t,x) = (F_1(t,x), F_2(t,x), F_3(t,x), \ldots)$$

and

$$G(t,x) = (G_1(t,x), G_2(t,x), G_3(t,x), \ldots).$$

As we did before, we let

$$G_j(t,x) = S(F_j(t,x))$$

for all $j = 1,2,3,\ldots$ We note that

$$F_j(t,U^s x) = F_j(t+s,x),$$

which means that for any two given values
Which means that for any two given values t_1, t_2 of t, F_j (t_1,x) and F_j (t_2,x) are equimeasurable functions of x. Furthermore, due to the translation invariance of S, the function G_j (t,x) has the same property. We, indeed, have

$$G_j(t,U^s x) = S(F_j(t,U^s x)) = S(F_j(t+s,x))$$
$$= G_j(t+s,x)$$

Let now $F_j^a(t,x) = F_j(t,x)$ if $|t|<a$, $F_j^a(t,x)=0$

otherwise, and let

$$G_j^a(t,x) = S(F_j^a(t,x))$$

Since S is positive (i.e., its values are nonnegative functions) and sublinear, we have

$$G_j(t,x) = S(F_j) = S(F_j^{a+\varepsilon} + (F_j - F_j^{a+\varepsilon}))$$

$$\leq S(F_j^{a+\varepsilon}) + S(F_j - F_j^{a+\varepsilon})$$

and since $F_j - F_j^{a+\varepsilon}$ has supported in $|t|>a+\varepsilon$, and S is semilocal, the last term on the right vanishes for $|t|\leq a$ for ε sufficiently large, independently of a. Thus, we have $G_j \leq G_j^{a+\varepsilon}$ for $|t|\leq a$. Suppose that S satisfies a vector-valued strong L^p norm inequality. Since G_j $(0,x)$ and G_j (t,x) are equimeasurable functions of x, for all j= 1,2,3,.., we have

$$2\int_X \|G(0,x)\|_{l^r}^p\, dx = \frac{1}{a}\int_{|t|<a} dt \int_X \|G(t,x)\|_{l^r}^p\, dx$$

$$\leq \frac{1}{a}\int_{|t|<a} dt \int_X \left\|G^{a+\varepsilon}(t,x)\right\|_{l^r}^p\, dx$$

$$= \frac{1}{a}\int_X dx \int_{|t|<a} \left\|G^{a+\varepsilon}(t,x)\right\|_{l^r}^p\, dt$$

and since $SF_j^{a+\varepsilon} = G_j^{a+\varepsilon}$ we have

$$\int_{|t|<a} \left\|G^{a+\varepsilon}(t,x)\right\|_{l^r}^p\, dt \leq C_1^p \int \left\|F^{a+\varepsilon}(t,x)\right\|_{l^r}^p\, dt,$$

whence substituting above we obtain

$$2\int_X \|G(0,x)\|_{l^r}^p\, dx \leq \frac{1}{a} C_1^p \int_X dx \int \left\|F^{a+\varepsilon}(t,x)\right\|_{l^r}^p\, dt$$

and again, since F_j (0, x) and F_j (t, x) are equimeasurable, for all j= 1,2,3,..,the last integral is equal to

$$2(a+\varepsilon)\int_X \|F(0,x)\|_{l^r}^p\, dx$$

and

$$\int_X \|G(0,x)\|_{l^r}^p\, dx \leq \frac{1}{a}(a+\varepsilon) C_1^p \int_X \|F(0,x)\|_{l^r}^p\, dx.$$

Letting a tend to infinity, we prove the first part of our theorem.
Suppose now that S satisfies the vector-valued weak type (p,p) inequality. For any given λ >0, let E and be the set of points where $\|G(0,x)\|_{l^r} > \lambda$ and $\left\|G^{a+\varepsilon}(t,x)\right\|_{l^r} > \lambda$, respectively, and $\tilde{E}_y$ the intersection of $\tilde{E}$ with the set {(t, x): x=y}. Then, we have

$$2a|E| \leq |\tilde{E}| = \int_X |\tilde{E}_x|\, dx.$$

On the other hand, since S satisfies the vector-valued weak type (p,p) inequality, we have

$$|\tilde{E}_x| \leq \frac{C_2^p}{\lambda^p}\int \left\|F^{a+\varepsilon}(t,x)\right\|_{l^r}^p dt.$$

Using the above inequalities and the fact that F_j (0, x) and F_j (t, x) are equimeasurable for all j = 1,2,3,. we have

$$|E| \leq \frac{C_2^p}{\lambda^p}(a+\varepsilon)\int \|F(0,x)\|_{l^r}^p dx.$$

When we let a tend to α we find the desired result.

An Application

Let f be a function on $\mathbb{R}$, the Hardy–Littlewood maximal function Mf defined by

$$Mf(t) = \sup_{t \in I} \frac{1}{|I|} \int_I |f(y)|\, dy$$

Where I denoted an arbitrary interval in $\mathbb{R}$. Consider now, an ergodic measure preserving transformation τ acting on a probability space (X, β, μ) then $M^{\#}$ is the ergodic maximal function defined by

$$M^{\#} f(t) = \sup_n \frac{1}{n} \sum_{k=0}^{n-1} \left| f(\tau^k x) \right|$$

Where f is a function on X.
Recall the following result of Fefferman and Stein.[2]

Lemma 1

Let $f = (f_1, f_2, f_3, \ldots)$ *be* a sequence of functions on $\mathbb{R}$ and $1 < r, p < \alpha$. Then, there are constants $C_{r,p} > 0$ and $C_r > 0$ such that

(a) $$\left\| \left(\sum_{j=1}^{\infty} \left| Mf_j(\cdot) \right|^r \right)^{1/r} \right\|_p \leq C_{r,p} \left\| \left(\sum_{j=1}^{\infty} \left| f_j(\cdot) \right|^r \right)^{1/r} \right\|_p$$

(b)
$$\left| \left\{ t \in \mathbb{R} : \left(\sum_{j=1}^{\infty} \left| Mf_j(t) \right|^r \right)^{1/r} > \lambda \right\} \right| \leq \frac{C_r}{\lambda} \left\| \left(\sum_{j=1}^{\infty} \left| f_j(\cdot) \right|^r \right)^{1/r} \right\|_1$$

for all $\lambda > 0$.

When we apply Theorem 1 to Lemma 1, we find the following result:

Theorem 2

Let $f = (f_1, f_2, f_3, \ldots)$ be a sequence of functions on $\mathbb{R}$ and $1 < r, p < \alpha$. Then, there are constants $C_{r,p} > 0$ and $C_r > 0$ such that

(c) $$\left\| \left(\sum_{j=1}^{\infty} \left| M^{\#} f_j(\cdot) \right|^r \right)^{1/r} \right\|_p \leq C_{r,p} \left\| \left(\sum_{j=1}^{\infty} \left| f_j(\cdot) \right|^r \right)^{1/r} \right\|_p$$

$$\mu \left\{ x \in X : \left(\sum_{j=1}^{\infty} \left| M^{\#} f_j(x) \right|^r \right)^{1/r} > \lambda \right\}$$

(d) $$\leq \frac{C_r}{\lambda} \left\| \left(\sum_{j=1}^{\infty} \left| f_j(\cdot) \right|^r \right)^{1/r} \right\|_1$$

for all $\lambda > 0$.

Note that the constants $C_{r,p}$ and C_r in Theorem 2 are the same constants as in Lemma 1.

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