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The Philosophical Implications of Set Theory in Infinity

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ABSTRACT

What does "infinity" mean? In fact, there are mathematical, physical, and metaphysical definitions of this concept. This study will focus on the scripting of the three philosophical foundations of mathematics — formalism, intuitionism, and logic-ism — in set theory (Snapper, 1979). Various examples will be provided regarding the concept of infinity for these three schools of thought. However, none of them can prove whether there is an infinite set or the existence of infinity. As such, it forms the foundational crisis of mathematics. Further elaboration on these philosophies leads to ideas of actual, potential, and absolute boundlessness, which correspond to three basic definitions of infinity. This thesis aims to correspond these philosophies to Roger Penrose's three world philosophy, in hope of implying the quantum mind. By employing rational proof and set theory, there is a likely possibility of building a human-like AI computer. To the authors' best knowledge, this thesis is the first to employ set symbols which connects body, mind, and spirit. More specifically, this paper aims to become the mathematical basis for the construction of a quantum computer. Using the Basic Metaphor of Infinity, as well as cognitive mechanisms such as conceptual metaphors and aspects, one can fully appreciate the transfinite cardinals' beauty (Nũńez, 2005). Indeed, the three mathematical philosophies map well with the three types of infinities and further fit perfectly with the body, mind, and spirit. In such a case, we may recognize how our set theory can be applied elegantly behind through mapping. This further implies the portraiture for something endless is anthropomorphic in nature or the perceptions of healing. In simple terms, because there is a connection between art and mathematics through infinity, one can enjoy the beauty of boundlessness (Maor, 1986). In essence mathematics is the science of researching the limitless.

Key words: Infinity, Cantor's Theory, Logic-ism

INTRODUCTION

What is the concept of infinity? According to the Encyclopaedia Britannica, one may refer to infinity as the concept about something that is endless, unlimited, or without bounds. In 1657, the English mathematician John Wallis introduced the common symbol "∞"for infinity. One can categorize the concept according to three schools of thought:

1. Mathematically: The counting points in number based on a continuous line, or the size of counting numbers sequence such as 1, 2, and 3...
2. Physically: Whether the number of stars is infinite or whether the universe wills last forever
3. Metaphysically: The discussion about god

It is, therefore, important to begin investigating the term "infinity" with these definitions in mind. Moreover, in the 19th century, Georg Cantor proposed what he called "transfinite numbers" in set theory. This, in turn, caused a great dispute between himself and Leopold Kronecker. Thus, one needs to understand the philosophical implications of infinity for modern set theory.

LITERATURE REVIEW

When one tries to resolve the conflict between Cantor and Kronecker, one needs first to understand the arguments behind their views. The

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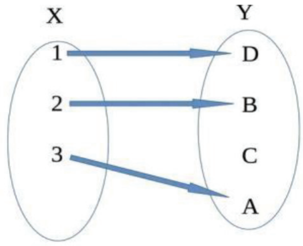
following sections will describe their differing perceptions on the foundations of mathematical philosophy concerning infinity.

Cantor's formalization of infinity

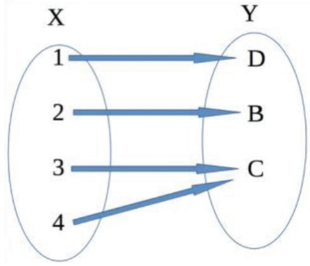
What is Cantor's mathematical definition of infinity? He developed the idea through set al- gebra and proposed what can be termed "infinity arithmetic." Indeed, Cantor earlier tried to for- malize his ideas that connected infinity and infinite sets. The origin of Cantor's ideas and their sub- sequent development will be explained in more depth in the following section.

The origins of cantor's theory — the paradox arisen from Galileo

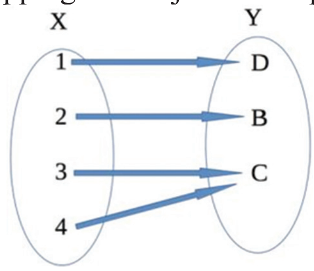
Galileo Galilei (1564–1642) had the role of acting as a physicist, mathematician, and astronomer.
He noticed that intuitively there are:
The same amount of natural numbers as the same amount in perfect squares of natural numbers (Velickovic, 2010).
One can then draw a one-to-one correspondence between them, for example:
"1" is assigned to 1^2,
"2" is assigned to $2^2 = 4$,
"3" is assigned to $3^2 = 9$,
It should be noted that the meaning of a function f in one-to-one is when $u_1 \neq u_2 \Rightarrow f(u_1) \neq f(u_2)$;
However, there may exist $v \in V$, but one cannot find $u \in U$ and $f(u) = v$. Therefore, the number of U ≤ the number of V.
At the same time, it is obviously that the set which consists of only natural square numbers is properly contained in the natural number set[1] (Velickovic, 2010). That is, the number of V ≤ the number of U.
Thus, one may conclude that the number of U = number of V
There must be a one-to-one and onto mapping (i.e., bijection) between the above two sets of numbers. To be more precise, one will have the following definition (as described by Velickovic, 2010):
Definition: When there are any sets X and Y, $X \preccurlyeq Y$, *if* there exists an injection from X into Y; $X \approx Y$ *when* it is both one-to-one and onto (i.e., bijection) mapping between X and Y.



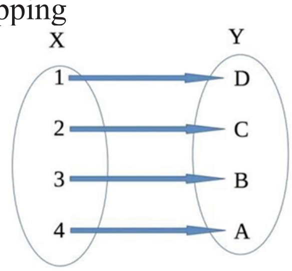
$X \preccurlyeq Y$ Surjective mapping $X \approx Y$



Injective Mapping Surjective Mapping



Bijective mapping



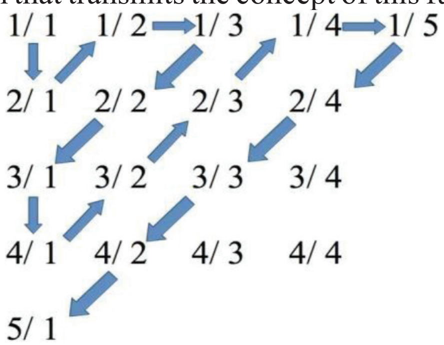
This leads to the following theorem:

Theorem (Cantor-Bernstein)
Given the condition $U \preccurlyeq V$and $V \preccurlyeq Y$*, it is suffices to show* $U \approx V$.
A brief proof (Rao, 2009) is as follows: This means that there are both injection mappings from U to V from V to U. Hence, by definition, for every element in V, there must be no more than one pre-image element in U such that f (u) = v, i.e., | U | ≤ in | V | where | | means the number of objects in the set U or V.
Similarly, for every element in U, there must be no more than one pre-image element in V such that g(v) = u, i.e.,| V | ≤ | U | where | | means the number of objects in the set U or V.
Thus, the | U | = | V | or $U \approx V$. This implies a bijection mapping exists between U and V. Details of the proof is shown as follows.

Proposition: X is infinite iff$X \approx X/\{x\}$, for any $x \in X$.
Definition: U is countable if U≈ℕ.
The author notes that a set U is said to be countable. The condition is a one-to-one and onto mapping exists between U and nature number set (or some subset of nature number). Thus, one may conclude that any finite set U is countable (when a 1-1 and onto mapping exists between U and the subset of {0, 1, 2 … p-1}. |U|= p where | | is the size of the set) (Rao, 2009).

Cantor's transfinite numbers

As previously mentioned, one can determine the count ability of the set indirectly through the size of a set. Thus, one has the following circumscriptions:
Definition: The size of a set X is defined as the cardinality. It is also denoted as |X|, where the cardinal is called the transfinite number.
Definition (Bulatov, 2012): | ℕ| is denoted by $\aleph_\circ$.
Theorem (Cantor): The set of rational numbers ℚis countable.
Proof (Schechter, 2002, Rao, 2009; Velickovic, 2010):
It is obvious that the nature number is contained in the rational number. Then, the size of the nature number is smaller than the size of rational number. Therefore, our goal is to prove the rational number is the subset of the nature number or the size of rational number is smaller than the size of nature number.
One may erect a *f*: ℚ→ ℕwith a spiral ladder diagram that transmits the concept of this function:



One may further construct the following ordered pairs:
(1, 1), (2, 1), (1, 2), (1, 3), (2, 2), (3, 1)…and then write each pair in a fractional form,
1/1, 2/1, 1/2, 1/3, 2/2, 3/1…, before deleting those repeated pairs and mapping them onto 1, 2, 3, 4, 5… as follows:
1/1, 2/1, 1/2, 1/3, 3/1…
↓ ↓ ↓ ↓ ↓
1 2 3 4 5…

Obviously, the above mapping is an injection between ℚand ℕand hence | ℚ|≤ | ℕ|. One will have ℚ≼ℕ. Therefore, with theorem, one will obtain ℚ≈ℕ, and by definition 2.1.1.4, ℚis countable. This seems straightforward but the violation found was so astonishing that Cantor said, "I see this result, but just cannot believe it!" (Maor, 1986: p.83)
Thus, the members of the set of fractions of collections are dense, but the number of members and the difference in the distance of natural numbers are equal (Maor, 1986). Studying this, Cantor decided to give a label to the collection of countable numbers, which he called the power of $\aleph_\circ$(here ℵ is pronounced as "aleph," the first Hebrew letter). Collections of the power of $\aleph_\circ$and natural numbers have the same number, so they are countable (Maor, 1986).
In addition, given the above proof, one may ask (as did Cantor): "Are all infinite sets countable?" (Maor, 1986: p.84) To answer this question, this study suggests using the following theorem below:
Theorem (Cantor): The set of real numbers ℝ is uncountable.
Proof: (Rao, 2009, Maor, 1986) On the contrary, suppose there is a bijection *f*: ℕ→ ℝ[0, 1]. One can enumerate the infinite list in general as follows:
$r_1 = 0.a_1a_2a_3 \ldots$
$r_2 = 0.b_1b_2b_3 \ldots$
$r_3 = 0.c_1c_2c_3 \ldots$
Where a_i, b_i, c_i are selected from ℕ.
Then, one may choose a diagonal such that it forms a new real number $r = 0.a_1b_2c_3 \ldots$. Now select the real number "t" and started replacing every digit of r, such as modifying each digit
y_i with $(y_i+2) \bmod 10$
This research states that the above infinite list of real numbers does not contain "t." On the contrary, one may assume "t" existed and it was just the k^{th} number as listed in the above. Then, one may observe that the difference between r and t lays in the k^{th} digit y_k such that the
(k^{th} digit) of t = the (k^{th} digit of r plus 2) mod 10.
That is, $t \neq r_k$ for any r_k in [0, 1]. Hence, one can conclude "t" is indeed a real number but obviously "t" cannot be found in the range of the proposed function *f* or [0, 1]. This is a contradiction to the fact that *f* must be a bijective function. Therefore, ℝ is uncountable. Cantor used the letter "C" to

represent the powerא for a set of rational numbers (Maor, 1986).

In this way, Cantor developed hierarchical orders for infinity. For example, the power ℵ denoted as C in such a set is higher than the hierarchy of the set with power $\aleph_\circ$. However, there will not be an infinite set with a hierarchy of power lower than $\aleph_\circ$; even if one tried to form a set of $\mathbb{N}^2$ and deleted all the non-$\mathbb{N}^2$, the result would remain the same (Maor, 1986).

The arithmetic of transfinite cardinal numbers

Consider the subsets of a set S = {a, b, c}: one can have {a}, {b}, {c}, {a, b}, {a, c}, and {b, c} together with an empty set { } and the original set {a, b, c} (Maor, 1986). In total, there are eight subsets of the set that contain three elements. One call the collection of these subsets the power set P(S). There are 2^3 such subsets. The result can be extended to any finite set that consists of n elements and that has 2^n elements for the power set P(N). Similarly, Cantor expanded the consequence to infinite sets. When one considers the power set P (inf) of any infinite set, the number of elements of such a set P(inf) is greater than that of the original set. This way of thinking was revolutionary and unbelievable in Cantor's time (Maor, 1986).

Cantor concluded: If one starts from any finite or infinite set S, one can create a new set P(S) that has more elements than set S (Maor, 1986). By repeating this process, one can make a newer power set. Thus, one can have infinitely many hierarchical infinite sets, and each new power set introduces a larger new set with larger ℵ. Cantor used 4 ,2$_\circ$א where in the hierarchy, those sets with the same cardinal number are those with one-to-one correspondence, while those sets with different numbers cannot be one-to-one (Maor, 1986).

The cardinal number $\aleph_\circ$,

$$_\circ 2^2)^{\aleph_\circ}\ 2^{\aleph)}$$

Cantor named these transfinite cardinal numbers. In the case of a finite set, the hierarchical relationship is as follows: $n < 2^n < (2^2)^n < \ldots$

When one expands these transfinite cardinal numbers, one will have

$$<_\circ 2^2)^{\aleph_\circ} > \ldots 2^{\aleph_\circ} > א^{)}$$

Hence, by extending the above comparison, Cantor established the so-called arithmetic of transfinite cardinal numbers that has some strange rules:

$1 + \aleph 0 = \aleph 0$

$\aleph 0 + \aleph 0 = \aleph 0$

$\aleph 0 * \aleph 0 = \aleph 0$

To conclude, Cantor tried to develop those philosophies concerning infinite sets with different sizes.[2] Nevertheless, his formalism caused a great deal of controversy with Kronecker's finitism — an extreme case for intuitionism. This will be discussed more in depth in the next section.

Kronecker's finitism and intuitive objections to transfinite numbers

Kronecker's finitism is a form of mathematical philosophy. The philosophy will accept only finite mathematical objects (neglect those infinite ones) as its major consideration. Therefore, the best way of understanding it is to compare the philosophy with the conventional philosophy of mathematics where there are some infinite mathematical objects (e.g., infinite sets) to be considered whenever reasonably. The major idea of the field finitistic mathematics asserts the non-existence of infinite objects such as infinite sets. In such case, all-natural numbers are thus existing. On the other hand, one may think the set of all-natural numbers to be a non-existent mathematical object. Therefore, one cannot quantify any infinite domains in a relevant way. One of the most famous mathematical theories in finitism is Skolem's primitive recursive arithmetic.[3]

Indeed, one can prove the non-existence of infinite sets by the following counter-example:

Consider a set S = {66454517, 3, and 507}. One can rewrite it as 66454517^7 03 507

1. Construct a number from a diagonal by taking the "units" digit from the first number, that is, "7" in this case
2. Next, one obtains the "tens" digit from the following number by adding a zero before "3"
3. Continue the recursive process until one obtains "507," which is just in the set S
4. The last step is to change each digit in the "found" number to any other digit. For example, 5 changes to 9, 0 changes to 1 and 7 could also change to 1
5. The result is "911."

However, the number is not in the set. This counter-example tells us that one cannot have a complete set of ALL-natural numbers as one will always find some numbers not in the set. Cantor's proof assumes that "infinitely many" is a valid idea, and it is employed in disproving the above counter-argument. According to Cantor, one can always create a number that is not in the set S but that has "infinitely many" digits. Hence, the created number is not a natural one and consequently must not be in the set S.

Even if one accepts that "infinitely many" is a valid idea, the counter-example is still true. This is because some digits created by the recursive process have a finite number of digit places away from the lowest-value digit. At the same time, other digits must have an infinite number of places away from the lowest-value digit. Obviously, there is an inconsistency that should not occur since the recursive process is uniform. Therefore, finitism demonstrates that Cantor's formalization of an infinite set is invalid and that only finite sets exist. These serious criticisms were one of the reasons why Cantor became gloomy (Maor, 1986). His former teacher, the famous mathematician Leopold Kronecker, tried to attack transfinite numbers even more. In fact, Kronecker was very conservative and not only resisted the concept of the infinite but also commented on those mathematical theories based on natural numbers. The reason for these attacks was not purely academic but also due to jealousy. He saw his student's reputation becoming greater than his own. Sadly, Cantor died in a mental hospital in 1918 (Maor, 1986).

To summarize, through Kronecker's finitism (the non-existence of infinite sets) and an intuitive recursive process, one can illustrate that there are no Cantor transfinite numbers and hence all the philosophies mentioned in section 2.1 fail. After reviewing the relation between infinite sets and formalism and intuitionism, this study will proceed to discuss Dedekind and logic-ism in the next section.

Dedekind's infinite set and logic-ism

According to Dedekind:

"A set X is infinite if only if it is equivalent to a proper subset of itself."[4]

Given sets S and T, they are said to be equivalent if and only if there exists a bijection f: S → T between elements of S and those of T.

Proof:

Case I: the "if"[4] part

Let X be an infinite set. Since every infinite set has a countably infinite subset, one can possibly construct one from X.

Suppose S = {a_1, a_2, a_3 …} is a countably infinite subset of X. One can create a partition of S into:

S_1 = {a_1, a_3, a_5 …} and S_2 = {a_2, a_4, a_6 …}

In addition, assume that there exists a bijection established between S and S_1 such that $a_n \leftrightarrow a_{n-1}$

One can further extend the bijection between $S\cup(\backslash XS=X$ and $S_1\cup(X\backslash S)=X\backslash S_2$

One can demonstrate that a bijection can be created between T and one of its proper subsets TS_2. Hence, one can conclude that if X is infinite, then it is equivalent to one of its proper subsets.

Case II: the "only if"[4] part

In contrast, suppose that X is equivalent to one of its proper subsets, say X_0,

that is, $X_0 \subset X$, and there exists a bijection f: X → X_0. Nevertheless, there is no bijection between the finite set and its proper subset. Therefore, X must be infinite.

Indeed, Dummett (1991, p.49) offered a critique of Dedekind's infinite set theory as follows:

In Dedekind's philosophy of mathematics, human mind can create mathematical objects freely. The idea, with his contemporaries shared widely, our minds can also operate and create abstract objects. This result can lead to a solipsistic mathematics conception; implicitly, each subject is entitled with this conception to feel assurance. Actually, he creates the mathematical objects by means of his own mental operations. These objects will finally coincide with its properties and what others have created through analogous operations. For Frege, such an assurance was not foundational: for him, we have the subjective content in our minds. The reason is one cannot find the way to compare them; one cannot know whether his idea is the same as others.

After reviewing the three categories of philosophy, one finds that none of them can fully explain the concept of infinity in mathematics. Further work should be done by mathematicians and philosophers to create a model for the concept (Lam, 2016). However, one should still appreciate the beauty in the method of the various proofs of infinity in these schools. In addition to these proofs, there are also daily examples of teaching infinity through mathematical philosophy; these will be discussed in detail in the next section.[1-10]

Examples of teaching infinity using mathematical philosophy

During every day learning at university, the present author has encountered several different concepts of infinity in mathematics. These will now be discussed one by one by applying mathematical philosophy.

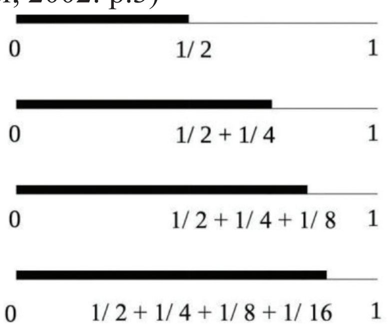
Teaching calculus using the concept of a limit in formalism

One of the most famous examples of teaching the idea of infinity is calculus. It allows one to formally teach the concept of a limit. First, one should understand the well-known paradoxes of motion:

The tortoise has a head start of about 100 m in front of Achilles, for example. Suppose that each of the racers begins running with some steady different speed (one moves very quick while another one is very slow). For some amount of countable time, Achilles runs to 100 m in which it is just the tortoise's early beginning. In this period, the tortoise has a displacement of only 10 m. Obviously Achilles needs some more time to run through this 10 m. After that, suppose the tortoise will displace further apart from the previous passing point at 110 m. Then Achilles requires some more time before reaching the third point, while at the same time, the tortoise is still moving slowly ahead to the forth point. Thus, whenever and wherever Achilles reaches the tortoise that it has already been, he always has to go farther. Therefore, since there are an infinite number of points, Achilles must follow the tortoise afterwards that it has stayed the moment before. Conclusively, he can never get over the tortoise.

Long before Cantor, Zeno's paradoxes (Zeno of Elea, 490–425 BC) showed how poor the people's understanding was of the concept of infinity:

"There is no motion, because to get anywhere you'd first have to get halfway, and before that you'd have to get a quarter of the way, etc." (Schechter, 2002: p.3)



The figure above shows one of Zeno's Paradoxes (Maor, 1986, p.34). It shows how much worse the knowledge of infinity was before Cantor.

At present, one can explain Zeno's Paradox as follows:

1/2 + 1/4 + 1/8 + 1/16 + 1/32 +… = 1, which is an infinite series with the sum equal to the limit of the finite partial sums.

Indeed, one will never get to 1. The infinite series should be rewritten as:

… + 1/32 + 1/16 + 1/8 + 1/4 + 1/2 = 1

This means that one will have infinitely many steps before one reaches 1/2 of the way or 1/4 of the way. If each step takes 1 s, one will never get anywhere (Schechter, 2002).

Nevertheless, when one take an in-depth look at the infinite series, if there is a sufficient number of items with a common ratio equal to "1/2," then the sum of the series can be arbitrarily approached closer to 1 (Maor, 1986). Suppose that the runner has a constant speed "s," and then the time needed between two points is directly proportional to the distance travelled. Therefore, the total period required for the whole journey is the sum of the series multiplied by the constant "s."

Hence, only a limited time is used for the travelling and this solves the controversy. The ancient Greeks did not believe that infinite series converge on a limit (Maor, 1986).

In reality, for the sequence 1, 1/2, 1/4… 1/n, the limit is zero (Maor, 1986). As n increases, the value in the sequence decreases and tends towards zero but will never equal zero. This means that if one has a sufficiently large n, the sequence a_n can arbitrarily approach close to zero. For example, if one needs a_n to be smaller than one in a thousand, then one requires n to be larger than one thousand. Similarly, one can even make a_n as close as one in ten billion (Maor, 1986).

However, mathematicians did not like this kind of a lengthy explanation. They preferred a simpler description (Maor, 1986):

Thus, one can represent the sequence by 1/nnd of a leng∞hus,lim $1/n = 0.n \to \infty$.

The term $\lim a_n$ is a kind of symbol under mathematical rules meaning the "Limiting value of…" when n tends to infinity. It clearly connects the ideas of endlessness and formalism. To be more precise, an abstract definition of a limit may be as follows:

Suppose there exists a finite value "a," then for any given positive integer ε, there is always a positive integer N(ε) such that n > N, $|x_n – a| < \varepsilon$; then one says the sequence $\{x_n\}$ tends to a limit "a;"

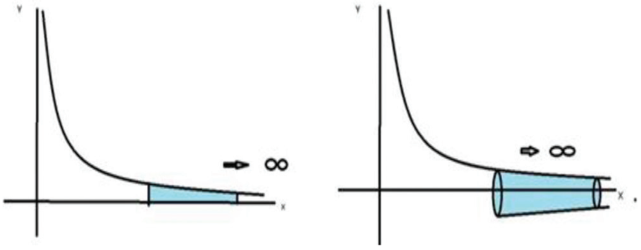
$\lim x_n = a$; $x_n \to a$ (Fudan, 1978)

n 1

After reviewing the example concerning the origins of linking and teaching the concept of a limit (a kind of an infinity, as well as calculus) and formalism, one may proceed to the second part.

Using geometry to learn infinity mathematics intuitively[11-14]

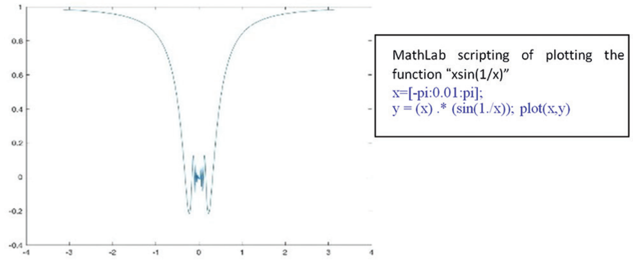
Consider the following function (Maor, 1986): *y = 1/x,* where the diagram is likely described as follows:



The above figure on the left shows a hyperbola (positive branch). Suppose it rotates around the x- axis; then it becomes a hyperbolic rotator, as shown on the right. If one integrates the surface area from *x* = 1 to the infinity of the rotator, the area becomes infinitely large. To be more precise, one can show that when calculating the corresponding surface area from *x* = 1 to some value >1 such as t and letting "t" tend to infinity, there will be no bounds on the area. However, the corresponding rotator's volume will tend to a certain limited value. In other words, there is a limit on the volume for the infinite 3D figure. If one wants to paint on the surface of the 3D figure, the job cannot be finished, as one needs an infinite amount of paint. On the other hand, if one paints the surface inside the rotator, then it is finite. This paradox cannot find a simple explanation and tells us that when anything is related to the concept of infinity, one's intuition can produce errors (Maor, 1986).

Furthermore, relationships exist between these paradoxes and the so-called "morbid functions." For example, if one considers *y = sin(1/x),* which has some special properties (Maor, 1986), as follows:

1. When *x* approaches zero, the graph of the function's oscillation becomes larger; thus, it can never be depicted completely. If one compares it with the hyperbola's graph, there is also a "discontinuity" when *x* = 0. However, the difference is that this function's graph will not tend to infinity, but only the oscillation frequency becomes infinite
2. If one considers the related function *y = xsin(1/x)*, the singular point x = 0 will disappear and hence the function becomes continuous.



$$f(x) = 1 \text{ when } x \in_{-1}^{\mathbb{Q}} \begin{cases} \\ -1 \text{ when } x \notin \mathbb{Q} \end{cases}$$

Then, for the upper line y = 1, there is an uncountable number of holes since ℝ is uncountable. On the other hand, for y = −1, there are a countable number of holes, as ℚ is countable. This special function discontinues everywhere and leads to controversies. As a result, the concept of "continuous" is redefined (Maor, 1986).[15-20]

To conclude, as stated by Jones *et al.* (2004: p.5), one may define "geometrical intuition" as a kind of skill to develop and compute geometrical figures the process of our mind. To see geometrical properties, one may relate images to different concepts and theorems in the field of geometry, and also decide where to start during the steps in solving geometrical problems. Thus, we highly recommend teachers should make use of the tasks that require students to imagine, manipulate geometrical figures as the result to link these geometrical intuition more directly with geometrical theory and finally involve the active use of imagination skills.

Applying contradictory logic-ism to studying infinity algebra

When one is concerned with contradictions in infinity, one might refer to an analogy proposed by the philosopher William Lane Craig who was involved in the calculations of certain kinds of infinite sets: "Imagine that one has an infinite number of marbles in his possession and he wants to give others some of them. In addition, suppose one (say A) wants to give the other one (say B) an infinite number of marbles." (Sewell, 2010: p.18)

Case I: One (A) could do that by giving all marbles to the other (B). Hence, A will have zero mar- bles left for himself or herself, i.e., ℵ0 - ℵ0 = 0.

Case II: Alternatively, A gives B all the odd numbered marbles. Then, A will still have an infinite number left for himself or herself, and B will also have an infinite set, that is, ℵ0 - ℵ0 = ℵ0. Obviously, the above two cases show contradictory results (0 and $\aleph_0$). However, subtraction and division of sets of equal amounts should not produce them. These collisions cast doubt on the idea that the infinite can be treated as a coherent notion (Sewell, 2010). Nevertheless, other philosophers such as Morrisionand Gumiski believe that there are discrepancies between subtracting infinite sets in transfinite mathematics (which will normally result in absurdities). The "removal" of one infinite set from another does not happen in the present real world (Sewell, 2010). In addition, the present author notes that the paradox leads to the logical contradiction of an infinite set being both "divisible" and yet "not divisible," which results in both a mathematical and a logical contradiction. There cannot be an infinite set of marbles or an infinite set of anything (Sewell, 2010). Thus, this may lead to the concept of indefiniteness, which can resolve the problem. After discussing the views and examples from the three categories of philosophy of mathematical infinity, one turns to the philosophical implications.

DISCUSSION — TYPES OF INFINITY AND THEIR PHILOSOPHICAL IMPLICATIONS

For each kind of mathematical philosophy, there are matters of corresponding philosophical significance such as the count ability of numbers, the start of the universe and the big bang, as well as the existence of God.[21]

Cantor's theory implies actual limitlessness (or the body) — the count ability of numbers

What is actual limitlessness? One usually refers to the concept as an ongoing process that is repeated over and over, but it is conceived as being "completed" or as having a final resultant state (Núñez, 2005). For instance, one can contemplate the sequence of regular polygons with an increasing number of sides where the distance from the center to any of the vertices remains constant. Indeed, one always begins with a triangle, then a square, a pentagon, a hexagon, and so on endlessly until the process changes a triangle into a circle. This is because after each iteration; there is an increase in one in the number of sides: the side's length decreases, but the distance "r" between the center of the polygon and the vertices remains the same. As one continues the process, the area and the perimeter of the polygon increasingly approach closer to the value of πr^2 and $2\pi r$, respectively. The circle has all the prototypical properties that circles must have, but conceptually it is a polygon. "The main theme of Cantor's Grundlagen is that there are multiple actual infinities because there is a realm of an actual, but increasable infinite known as the transfinite." (Newstead, 2009: p.536). This is the reason why it is proposed here that there is a connection between actual limitlessness and Cantor's theory of transfinite numbers. Through these cardinals, one can tell the count ability of "natural," "rational," and even "real" numbers as mentioned earlier.

Kronecker's disagreement suggests potential boundlessness (or the mind) — big bang theory

In contrast, potential boundlessness means "a non-terminating process (such as "add 1 to the previous number") produces an unending "infinite" sequence of results, but each individual result is finite and is achieved in a finite number of steps." It is proposed here that this definition is related to Kronecker's finitism as well as Skolem's primitive recursive arithmetic. In fact, according to Craig, the past is made up of a series of events (E_1, E_2…E_n). Usually, one refers to "the past" as the set of all these events. Hence, the past cannot be an actual infinity. If one begins with a single event and adds one another gradually, one will never obtain an actual infinite number of events. What one ends up with is a set whose number of constituents becomes ever larger and tends toward infinity but never actually reaches it. This is defined as a "potential infinity." Therefore, it appears to the present author that there must be a beginning of time. What are entailed by "potential boundlessness" are the Big Bang theory and thus a "start of the universe."

Recently, some astronomers have combined observations and mathematical models to develop a workable theory of how the universe came to be.[14] According to their theory, our universe previously existed as a "singularity" 13.7 billion years ago.[16] Singularities are thought to be "black holes" with intense gravitational pressure where finite matter is crushed into infinite density. Our universe is believed to have started as an infinitely hot, infinitesimally small, infinitely dense, something — a singularity. Beyond the initial shape, the singularity seemingly inflated (the so-called "Big Bang"), cooled and expanded, and began changing from being extremely small and extremely hot to the current temperature and size of the universe.

The process is ongoing, and human beings are a part of it — incredible creatures that live on a unique planet, circling a beautiful star. The planet is also clustered with some hundred billion other stars in a galaxy soaring through the cosmos. All this is happening inside an expanding universe that began as an infinitesimal singularity that appeared out of nowhere and for reasons unknown. This is one suggestion of what the Big Bang theory is.[16]

Dedekind's concept describes absolute endlessness (or the spirit) — the existence of god

What is the definition of absolute endlessness? Some philosophers believe that the Class V of all sets, the mindscape, the cosmos, and God are all examples of what "absolute" is. Indeed, one may define the term "absolute" in the sense of "non-relative, non-subjective."[17] The absolute itself exists together with the highest form of completeness. In fact, there is a relationship between the limitlessness of God and mathematical infinity. As St. Gregory said, "No matter how far our mind may have progressed in the contemplation of God, it does not attain to what He is, but to what is beneath Him." (Rucker, 2013: p.44) In such a case, one is now at the starting point of the infinite dialectic process since one is trying to establish an image of the whole mindscape.

The dialectic process happens in the following way:

1. First, one collects a group of thoughts into a single thought T
2. When one is in a conscious state of mind T, a new thought is then constructed that one has not yet accounted for previously
3. One's mindscape is improved in terms of thought, including the elements of T plus T itself.

In mathematical symbolic terms, one may consider the *n*th thought T_n that one can define inductively as follows: $T_o = \emptyset$ and $T_{n+1} = T_n \cup \{T_n\}$, for any sets *A* and *B*. *A*U*B* means the set of all the sets that are members of *A* or *B*. Nevertheless, one may have another inductive definition: $T_n = \{T_m: m<n\}$, which means that "T_n is the set of all T_m such that *m* is less than *n*."[17] However, T plus "T" is not always different from the thought T. In the case of a mind M, it is already fully self-aware and therefore M plus "M" is no different from M., that is, M U {M} = M. Rucker (2013) discusses in detail "absolute" in terms of the rational and mystical. Indeed, in 1887, one of Cantor's friends, Richard Dedekind, published a proof and claimed that the mindscape is infinite. Dedekind's term for the mindscape was *Gedankenwelt,* which literally means "thought-world."

By continuing the repetitive process, Dedekind proved the infinitude of the mindscape:

{*s, s* is a possible thought, *s* is a possible thought is a possible thought,...}

Therefore, this shows that the class of all sets, the mindscape and the class of all true propositions, are all infinite. In similar terms, one can tell whether God is endless. There are comments about.

God's existence in Rucker's work; however, this author believes in the existence of God and his being "absolute."

To sum up, the three mathematical philosophies map well with the three types of infinities and further fit perfectly with the body, mind and spirit. In such a case, we may recognize how our set theory can be applied elegantly behind through mapping. This further implies the portraiture for something endless is anthropomorphic in nature or the perceptions of healing (Kusilka, 2014). The theory of perception healing may thus be arisen (Bedford, 2012). However, this is out of the scope of the present discussion.

CONCLUSIONS — INFINITY ACTS AS A LINK BETWEEN ART AND MATHEMATICS

After reviewing, providing examples and discussing infinity, the question remains, "What is the role of limitlessness?" This study finds that it acts as a connection between art and mathematics. For example, one can appreciate the beauty of transfinite cardinals and hence imply that the portrait of infinity has a human face (Nũńez, 2005):

1. Aspectual systems consist of continuative and iterative processes, ideal and non-ideal structures with a starting and completion status, etc.
2. Conceptual metaphors, as in the case of Cantor's metaphor that the same number is pair ability

3. Conceptual blending, such as the use of multiple implicit basic mappings of infinity in Cantor's proofs

Indeed, Cantor's theories and proofs often worked against mainstream mathematical thought and that is why he surprised many people (Carey, 2005). His idea also caused discomfort among certain mathematics professionals. As Meschkowski so eloquently wrote, "Cantor's theorem is thus a beautiful example of a mathematical paradox, of a true statement which seems to be false to the uninformed" (quoted. in Dauben, 1979).

Nevertheless, what a mathematician focuses on with infinity is that "Mathematics takes us into the region of absolute necessity to which not only the actual world, but every possible world, must conform." (Egner *et al*., 2009: p. 229).

From a philosopher's point of view, "Mathematics is an ideal world and an eternal edifice of truth… In the contemplation of its serene beauty man can find refuge from the world full of evil and suffering."(Copleston, 1966: p. 438).

According to the astronomer James Jeans (1877–1946), "God is a mathematician" and most mathematicians say "God made the numbers. All the rest is made by human." (Peat, 2009: p.28).

In a nutshell, it is suggested here that mathematics is the science of studying infinity. When one discusses its daily applications, these may include calculating the gravitational force of infinite mass, determining whether infinity exists in our physical universe, making infinite regression arguments, computing arithmetic overflow, dividing by zero, etc.

The poem's study found the following:

"This lonely hill was always dear to me, and this hedgerow, which cuts off the view of so much of the last horizon. But sitting here and gazing, I can see beyond, in my mind's eye unending spaces, and superhuman silences, and depthless calm, till what I feels almost fear. And when I hear the wind stir in these branches, I begin comparing that endless stillness with this noise: And the eternal comes to mind, and the dead seasons, and the present living one, and how it sounds. So my mind sinks in this immensity: and foundering is sweet in such a sea" (Giacomo, 1819: p. 106).

Limitations

There are limitations to this thesis. They are as follows:

Cantor's theory may not be true, but it can be modified so that it contains no mistakes. One of the defects in Cantor's theory is Russell's paradox, as follows:

"If S were the set of all sets then $\boldsymbol{P}(S)$ would at the same time be bigger than S and a subset of S."

To avoid such a paradox, one may extend set theory into the 'New

Foundation' (NF) set theory. Instead of assuming a set, we consider $\{s \in S: s \notin f(s)\}$ as a local 'type theory' and as a set in NF. One can then easily show by proof of contradiction that

$$|P(S)| \quad |P(S)|$$

and eliminate Russell's Paradox.

Since my special "rationalization" is based on the assumptions of both set theory and logic, it may fail if one challenges the validity of the premises. This is the question of Platonism and anti- Platonism. Another main result of Cantor's theory is the famous "Continuum Hypothesis" problem: Does any set exist that has a cardinality between a natural number and a real number?

One may even extend the above problem as follows: Does any set exist that has a size between $|S|$ and $|\boldsymbol{P}(S)|$ for some infinite S? that is, the Generalized Continuum Hypothesis problem.

This author's answer is that there are various mathematical structures that exist between natural and real numbers. One may sub-divide them into as small parts as possible. Their cardinality converges to either card (N) or card (R). The problem of the continuum hypothesis is just our analogue world, which is continuous (or our real and human world). However, the digital world is only "0" — card (N) and "1" — card(R). At the same time, one can make the analog one as detailed as one wants since there are structures between N and R.

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