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On a Class of Functions Analytic in a Half Plane

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ABSTRACT

In this paper, we have defined a class of functions represented by generalized Dirichlet series whose coefficients satisfy certain given conditions. Region of convergence is obtained depending on the fixed Dirichlet series. This class is complete and becomes an Hilbert space with respect to the inner product defined. Schauder basis is also obtained.

Key words: Abscissa of convergence, abscissa of absolute convergence, Dirichlet series, inner product, norm, Schauder basis

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INTRODUCTION

In the present work, we shall throughout consider the functions `f' represented by generalized Dirichlet series $f(s)=\sum_{k=1}^{\infty} a_k e^{s\lambda_k}$ with positive exponents rather than negative ones. It is just to have an analogy with the power series $\sum_{k=1}^{\infty} a_k z^k$. The only difference in the characteristics of the Dirichlet series with positive exponents, as compared to the same, with negative exponents lies in the region of their convergence that is what we call the half plane of convergence. The Dirichlet series with positive exponents converges in the left half plane whereas the one with negative exponents converges in the right half plane. We shall also take the liberty of interpreting the results of all those workers who have considered the Dirichlet series with negative exponents in our terminology.[1-5]

It is well known that the Dirichlet series[2,5] with positive exponents (frequencies), in its most generalized form, is given by

$$f(s)=\sum_{k=1}^{\infty} a_k e^{s\lambda_k} \tag{1.1}$$

Where, $s=\sigma+it$ (σ,t real variables), $\{a_k\}$ in general, is a complex sequence, $\{\lambda_k\}$ is a strictly increasing sequence of positive real numbers, and $\lambda_k\to\infty$ as $k\to\infty$. It has been proved in[2] that the exponents $\{\lambda_k\}$ satisfy

$$\limsup_{k\to\infty} \frac{\log k}{\lambda_k} = D < \infty \tag{1.2}$$

The abscissa of ordinary convergence (abs. convergence) σ_c (σ_a) of the Dirichlet series (1.1) is defined as the least upper bound of all those `σ' for which the series converges (abs.). These are given as follows

$$\sigma_c = \begin{cases} -\limsup_{k\to\infty} \dfrac{\log\left|\sum_{m=1}^{k} a_m\right|}{\lambda_k} & \text{provided} \\ & \sigma \le 0 \text{ or } \sum_{k=1}^{\infty} a_k \text{ diverges} \\ -\limsup_{k\to\infty} \dfrac{\log\left|\sum_{m=k+1}^{\infty} a_m\right|}{\lambda_k} & \text{provided} \\ & \sigma > 0 \text{ or } \sum_{k=1}^{\infty} a_k \text{ converges} \end{cases} \tag{1.3}$$

$$\sigma_a = \begin{cases} -\limsup_{k\to\infty} \dfrac{\log\left|\sum_{m=1}^{k} a_m\right|}{\lambda_k} : \sigma \le 0 \\ \text{or } \sum_{k=1}^{\infty} a_k \text{ diverges (abs.)} \\ -\limsup_{k\to\infty} \dfrac{\log\left|\sum_{m=k+1}^{\infty} a_m\right|}{\lambda_k} : \sigma > 0 \\ \text{or } \sum_{k=1}^{\infty} a_k \text{ converges (abs.)} \end{cases} \tag{1.4}$$

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Also

$$0 \le \sigma_c - \sigma_a \le D \tag{1.5}$$

If D given in Equation (1.2) is zero, then the abscissa of ordinary convergence, that is, σ_c and the abscissa of absolute convergence, that is, σ_a have the same value which is given by,

$$\sigma_c = \sigma_a = -\limsup_{k\to\infty} \frac{\log|\alpha_k|}{\lambda_k} \tag{1.6}$$

Let

$$u(s) = \sum_{k=1}^{\infty} \alpha_k e^{s\lambda_k} \tag{1.7}$$

be a fixed Dirichlet serieswith given exponents or frequencies $\{\lambda_k\}$ satisfying,

$$\limsup_{k\to\infty} \frac{\log k}{\lambda_k} = D < \infty \tag{1.8}$$

and the coefficients $\{\alpha_k\}$ being a fixed sequence of non-zero complex numbers satisfying the condition,

$$-\limsup_{k\to\infty} \frac{\log|\alpha_k|}{\lambda_k} = A \tag{1.9}$$

Where, A is any arbitrary but fixed real number. If D is zero, then the Dirichlet series $u(s)$ has coincident abscissa of absolute convergence (σ_a^u) and abscissa of ordinary convergence (σ_c^u), which is given by the formula,

$$\sigma_c^u = \sigma_a^u = -\limsup_{k\to\infty} \frac{\log|\alpha_k|}{\lambda_k} = A \tag{1.10}$$

or equivalently by $-\limsup_{k\to\infty} |a_k|^{\frac{1}{\lambda_k}} = e^{-A}$.

The left plane $\sigma<A$ is denoted by R_u and is called the region of convergence of the fixed Dirichlet series $u(s)$. Furthermore, the series $u(s)$ converges uniformly in each left half plane $\sigma=A-\epsilon$, $\epsilon>0$. The sum of the series $u(s)$ is an analytic function in the left half plane $\sigma<A$ and an entire function if $A=+\infty$. The coefficients of the fixed Dirichlet series can be expressed in terms of its sum function by the Hadamard formula,

$$\alpha_k = \lim_{T\to\circ} \frac{1}{2T} \int_{-T}^{T} u(\sigma' + it') e^{-\lambda_k (A'+it)} dt$$

Let the class $\Omega(u,p)$ is defined as follows,

$\Omega(u,p)=\{f : f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}$ and

$$\sum_{k=1}^{\infty} \left|\frac{a_k}{\pm_k}\right|^p < \infty \ for\ 1 \le p < \infty\}$$

It can easily be seen that the function `u' given by the fixed Dirichlet series $u(s)$ does not belong to $\Omega(u,p)$ for $p\ge 1$.

Now, for $f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}$ and

$$g,\ g(s) = \sum_{k=1}^{\infty} b_k e^{s\lambda_k} \in \&(u,p)$$

We define the following pointwise linear operations and norm in $\Omega(u,p)$ as in the space $\Omega_{u,c}$) in the following way.

a. $(f + g)(s) = \sum_{k=1}^{\infty} (a_k + b_k) e^{s\lambda_k}$

b. $(\mu f)(s) = \sum_{k=1}^{\infty} \mu a_k e^{s\lambda_k}$ where μ is a scalar

c. $\|f\|_p^p = \sum_{k=1}^{\infty} \left|\frac{a_k}{\pm_k}\right|^p$

Clearly, $\|f\|_p^p$ exists. It can further be seen that $\Omega(u,p)$ is a normed linear space also.

Ω(*u,p*) As a Banach space[2]

Here, we have proved that $\Omega(u,p)$ is a Banach space for $1\le p<\infty$ which cannot become a Hilbert space unless $p=2$ and possess the property of uniform convergence[3] over every compact subset in the region of convergence R_u of u.

Theorem [1]:

$\Omega(u,p)$ is a Banach space for $p\ge 1$.

Proof:

Let $\{f_l\}$ be a Cauchy sequence in $\Omega(u,p)$ such that

$f_l(s) = \sum_{k=1}^{\infty} a_{lk} e^{s\lambda_k}$ then

$$\sum_{k=1}^{\circ} \left|\frac{a_{lk}}{\pm_k}\right|^p < \infty k$$

Since, $\{f_l\}$ is a Cauchy sequence in $\Omega(u,p)$ so, for a given $\in>0$, there exists a positive integer $N_0(\in)$ such that

$$\|f_l - f_m\|_p < \in \text{ for } l, m > N_0(\in)$$

or $\left(\sum_{k=1}^{\infty} \left|\frac{a_{lk} - a_{mk}}{\alpha_k}\right|^p\right)^{\frac{1}{p}} < \in$

$$\Rightarrow \sum_{k=1}^{\infty} \left|\frac{a_{lk} - a_{mk}}{\alpha_k}\right|^p < \in^p \tag{2.1}$$

$$\Rightarrow \left|\frac{a_{lk} - a_{mk}}{\alpha_k}\right|^p < \in^p \text{ for each } k$$

$\Rightarrow \left|\frac{a_{lk-}a_{mk}}{\alpha_k}\right| < \in$ for each k

This shows that $\left\{\frac{a_{lk}}{\alpha_k}\right\}$ is a Cauchy sequence in C for each k and so converges to $\frac{a_{0k}}{\alpha_k}$ (say) as $l\to\infty$.

i.e., $\lim_{l\to\infty}\frac{a_{lk}}{\alpha_k}=\frac{a_{0k}}{\alpha_k}$

Define a $f_0(s)=\sum_{k=1}^{\infty}a_{0k}e^{s\lambda_k}$, we show that $f_l\to f_0$ as $l\to\infty$ and $f_0\in\Omega(u,p)$.

From Equation (2.1), we see that for any positive integer m, $\sum_{k=1}^{m}\left|\frac{a_{lk-}a_{mk}}{\alpha_k}\right|^p < \in^p$. Let $m\to\infty$ then $\sum_{k=1}^{\infty}\left|\frac{a_{lk-}a_{0k}}{\alpha_k}\right|^p < \in^p < \in$ for $l>N_0$ ($\in$). This shows that $(f_l\text{-}f_0)$ belongs to $\Omega(u,p)$ such that $\left\|f_l-f_{0p}\right\| < \in$ i.e. $f_l\to f_0$ as $\to\infty$. Now, it follows then $f_0=f_l+(f_0\text{-}f_l)\in\Omega(u,p)$.

Theorem [2]:

$f_l\to f$ in $\Omega(u,p)\Rightarrow f_l(s)\to f(s)$ uniformly over every compact subset of the region of convergence R_u of u.

Proof:

Let S be a compact subset in the region of convergence R_u, then we get a rectangle T in R_u containing S where

$T=\{(\sigma,t):\sigma_1\le\sigma\le\sigma_2,t_1\le t\le t_2\}$

Let

σ_3 satisfies $\sigma_2<\sigma_3<\sigma$ so that $\theta=\frac{e^{\sigma_2}}{e^{\sigma_3}}<1$. Then, for any given $\epsilon>0$ choose a `η' such that

$\eta.M\sum_{k=1}^{\infty}\theta^{\lambda_k}<\epsilon$ where $|\pm_k|e^{\lambda_k\sigma_3}\le M<\infty$

and

$\|f_l-f\|_p<\eta$ for $l\ge l_0(\eta)$

$$\Rightarrow \sum_{k=1}^{\infty}\left|\frac{a_{lk}-a_k}{\alpha_k}\right|^p \le \cdot^{\,p} \text{ for } l\ge l_0(\eta)$$

$$\left|\frac{a_{lk}-a_k}{\alpha_k}\right|^p<\eta^p \text{ for each } k=1,2,3,\ldots.\ l\ge l_0\ (\eta)$$

Then for $s\in S$ and $l\ge l_0$, we have

$$|f_l(s)-f(s)|\le\sum_{k=1}^{\infty}|a_{lk}-a_k|e^{\sigma\lambda_k}$$

$$\le\sum_{k=1}^{\infty}\left|\frac{a_{lk}-a_k}{\alpha_k}\right||\pm_k|e^{\sigma\lambda_k}$$

$$\le\eta\sum_{k=1}^{\infty}|\alpha_k|e^{\sigma_3\lambda_k}\le\eta M\sum_{k=1}^{\infty}\theta^{\lambda_k}<\varepsilon$$

which proves uniform convergence[3] over every compact subset of the region of convergence R_u of u.

Theorem [3]:

$\Omega(u,p)$ cannot become a Hilbert space unless $p=2$.

Proof:

Let $f(s)=\alpha_1e^{s\lambda_1}+\alpha_2e^{s\lambda_2}$, $g(s)=\alpha_1e^{s\lambda_1}+(-\alpha_2)e^{s\lambda_2}$. Clearly, $f,g\in\Omega(u,p)$.

For $p\neq 2$, we see that

$$f_p^p=\sum_{k=1}^{2}\left|\frac{a_k}{\pm_k}\right|^p=\left|\frac{\alpha_1}{\alpha_1}\right|^p+\left|\frac{\alpha_2}{\alpha_2}\right|^p \quad\Rightarrow f_p=2^{\frac{1}{p}},$$

similarly $g_p=2^{\frac{1}{p}}$

Also $\|f+g\|_p^p=\left|\frac{2\alpha_1}{\alpha_1}\right|^p=2^p\Rightarrow\|f+g\|_p=2$

and $\|f-g\|_p^p=\left|\frac{2\alpha_2}{\alpha_2}\right|^p=2^p\Rightarrow\|f-g\|_p=2$

Clearly $\|f+g\|_p^2+\|f-g\|_p^2\neq 2.\|f\|_p^2+2.\|g\|_p^2$ as $8\neq 2\left(2^{\frac{2}{p}}+2^{\frac{2}{p}}\right)$ for $p\neq 2$. Thus, Parallelogram law[1,6] does not hold showing that it is not a Hilbert space.

Next, we find Schauder basis[4] for $\Omega(u,p)$ $(1\le p<\infty)$ in the following Lemma.

Theorem [4]:

The set $\left\{\pm_k e^{s\lambda_k}\right\}_{k=1}^{\infty}$ is the Schauder basis for $\Omega(u,p)$.

Proof:

Let $f\in\ \Omega(u,p)$, where $f:f(s)=\sum_{k=1}^{\infty}a_ke^{s\lambda_k}=\sum_{k=1}^{\infty}a_k'\delta_k$ where $\delta_k(s)=\pm_ke^{s\lambda_k}$ and $a_k'=\frac{a_k}{\pm_k}$ fork=1,2,3. Then, for a given $\varepsilon>0$, there exists a positive integer `n' such that $\sum_{k=n+1}^{\infty}\left|\frac{a_k}{\pm_k}\right|^p<\varepsilon$

But then

$$\lim_{n\to\infty}\left\|f-\sum_{k=1}^{n}a_k'\delta_k\right\|_p^p=\lim_{n\to\infty}\left\|\sum_{k=n+1}^{\infty}a_k'\delta_k\right\|_p^p$$

$$=\lim_{n\to\infty}\sum_{k=n+1}^{\infty}\left|\frac{a_k}{\acute{a}_k}\right|^p=0$$

Thus $f=\sum_{k=1}^{\infty} a_k^{'}\delta_k \Rightarrow f(s)=\sum_{k=1}^{\infty} a_k^{'}\delta_k(s)$ (2.2)

Now, we will show the uniqueness of the above representation (2.2). If possible, let us assume $f=\sum_{k=1}^{\infty} b_k^{'}\delta_k$ then

$$\left\|\sum_{k=1}^{n}(a_k^{'}-b_k^{'})\delta_k\right\|_p^p \le$$

$$\left\|f-\sum_{k=1}^{n} a_k^{'}\delta_k\right\|_p^p+\left\|f-\sum_{k=1}^{n} b_k^{'}\delta_k\right\|_p^p$$

$\to 0$ as $n\to\infty$.

Therefore, $\sum_{k=1}^{n}\left|a_k^{'}-b_k^{'}\right|^p \to 0$ as $n\to\infty$.

Consequently, $a_k^{'}=b_k^{'}$ for each k. Hence, $a_k=b_k$ for each k.

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