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On Review of the Consistency and Stability Results for Linear Multistep Iteration Methods in the Solution of Uncoupled Systems of Initial Value Differential Problems

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ABSTRACT

The linear multistep method is a numerical method for solving the initial value problem, x' = f(t,x); $x(t_0) = x_0$. A typical linear multistep method is given by $X_{n+k} = \sum_{j=0}^{k-1} \alpha_j x_{n+j} + h \sum_{j=0}^{k} \beta_j f_{n+j}$; $k \geq 1$. If $\beta k \neq 0$, then, the method is called implicit. Otherwise, it is called an explicit method. Several methods abound for deriving linear multistep methods; however, in this work, we center on analysis of the convergence and stability of the linear multistep methods. To this effect, we discussed extensively on the convergence, relative, and weak stability theories while preliminarily, we discussed the truncation errors of the linear multistep methods and consistency conditions for the convergence of the linear multistep methods.

Key words: The linear multistep method, consistency condition, convergence condition, stability condition, relative stability condition and weak stability condition

INTRODUCTION

Preliminary Concepts

Consider the differential equation

$x = f(t,x); x(t_0) = x_0$ (1.1.1)

A computational method for determining the sequence $\{x_n\}$ that takes the form of a linear relationship between x_{n+j}, f_{n+j}, j = 0, 1,…, k is called the linear multistep method of step number k or k-step methods; $f_{n+j} = f(t_{n+j}, x_{n+j}) = f(t_{n+j}, x(t_{n+j}))$. The general linear multistep method[1] may be given thus –

$$\sum_{j0}^{k} \alpha_j x_{n+j} = h \sum_{j0}^{k} \beta_j f_{n+j} \quad (1.1.2)$$

Where, α_j and β_j are constants, $\alpha_k \neq 0$ and not both α_0 and β_0 are zeros. We may without loss of generality, and for the avoidance of arbitrariness set $\alpha_k = 1$ throughout

Suppose in Equation (1.1.2), $\beta_k = 0$,[2] then, it is called an explicit method since it yields the current value x_{n+k} directly in terms of x_{n+j}, f_{n+j}, j = 0, 1,…, k-1 which by this stage of computation have already been calculated. If however, $\beta_k \neq 0$, then Equation (1.1.2) is called an implicit method which requires the solution at each stage of the computation of the equation.

$$X_{n+k} = h\beta_k f(t_{n+k}, x_{n+k}) + g \quad (1.1.3)$$

Where, g is a known function of the previously calculated values, x_{n+j}, f_{n+j}, j = 0, 1,…, k-1. Suppose $\|f(t, x_1) - f(t, x_2\| \leq L \|x_1 - x_2\|$ then we set $M = \ell h|\beta_k|$ so that a unique solution for x_{n+k} exists and the computational iteration converges to x_{n+k} if

$$\ell h|\beta_k| < 1 \text{ i.e. } h < \frac{1}{\ell h|\beta_k|}$$

We, therefore, see that implicit methods call for a substantially greater deal of computational efforts than explicit methods; whereas, on the other hand, for a given step number k, implicit methods can be made more accurate than explicit ones. Moreover, they have favorable stability properties as will be seen in chapter three.

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Definition (1.1.1)[3,4]

The sequence of points $\{t_n\}$ given by $t_n = t_0 + nh$ ($t_0 = a$), n = 0, 1, 2,…, are called the mesh points while h (always a constant) is the mesh (step) length.

Definition (1.1.2)[3,4]

Let f:Ψ: IR→IR be functions. We say that f is big 0 of Ψ as $x \to x_0$ and write f(x) = 0 (Ψ (x)) as $x \to x_0$ if there exists k > 0 constant, such that

$$\lim_{X \to X_0} \left| \frac{f(x)}{\Psi(x)} \right| = k$$

Definition (1.1.3)[3,4]

Let f: Ψ; IR→IR be functions. We say that f is small 0 of Ψ as $x \to x_0$ and write f(x) = 0 (Ψ (x)) as $x \to x_0$ if

$$\lim_{X \to X_0} \left| \frac{f(x)}{\Psi(x)} \right| = 0$$

The Local Truncation Error

Let $x_{n+j} = x(t_{n+j})$, j = 0,1,2,…,k-1, if 0_{n+j} denotes the numerical solution with above exact values, then (Enright and Hall [1976])

$$x_{n+k} + \sum_{j=0}^{k-1} \alpha_j x_{n+j} = h\beta_k f(t_{n+k}, 0_{n+k}) + h\sum_{j=0}^{k-1} \beta_j f\left(t_{n+j}, x_{n+j}\right)$$

Localizing assumption means that no previous truncation errors have been made and that $x_{n+j} = x(t_{n+j})$, j = 0, 1,…,k-1 this implies that

$$T_{n+k} = \sum_{j=0}^{k-1} \alpha_j x\left(t_{n+j}\right) - h\sum_{j=0}^{k-1} \beta_j f\left(t_{n+j}, \left(x_{n+j}\right)\right)$$

Let us define the local truncation error as

$$x_{n+k} + \sum_{j=0}^{k} \alpha_j x\left(t_{n+j}\right) = h\beta_k f(t_{n+k}, x_{n+k}) + h\sum_{j=0}^{k} \beta_j f\left(t_{n+j}, \left(x_{n+j}\right)\right)$$

Subtracting we get

$$x\left(t_{n+k}\right) - 0_{n+k} = T_{n+k} + h\beta_k [f\left(t_{n+k}, x\left(t_{n+k}\right)\right) - f(t_{n+k}, 0_{n+k})]$$

Applying the mean value theorem (Brice *et al.* [1969])

Then

$$x\left(t_{n+k}\right) + \sum_{j=0}^{k} \alpha_j x\left(t_{n+j}\right) = T_{n+k} + h\beta_k f\left(t_{n+k}, x\left(t_{n+k}\right)\right) + h\sum_{j=0}^{k} \beta_j f\left(t_{n+j}, x\left(t_{n+j}\right)\right)$$

$$f(x_{n+k}, x\left(t_{n+k}\right) - f(t_{n+k}, 0_{n+k}) = (x\left(t_{n+k}\right) - 0_{n+k}) \left.\frac{\partial f}{\partial x}\right|_{t_{n+k};T_{n+k}}$$

Where η_{n+k} lies between 0_{n+k} and $x(t_{n+k})$
Therefore

$$\left(1 - h\beta_k \left.\frac{\partial f}{\partial x}\right|_{t_{n+k};\eta_{n+k}}\right)(x\left(t_{n+k}\right) - 0_{n+k}) = T_{n+k} \tag{1.2.2}$$

Let e_{n+k} represent the error at (n+k) point, so that if the method is explicit $\beta_k = 0$, then $T_{n+k} = e_{n+k}$ and if the method is implicit $\beta_k \neq 0$ and is small then

$$T_{n+k} \approx e_{n+k} h\beta_k \left.\left(\frac{\partial f}{\partial x}\right)\right|_{t_{n+k};\eta_{n+k}}$$

To proceed as shown below, we[5,6] note the following useful formula

$$X\left(t_{n+j}\right) = x\left(t_n + jh\right) = x\left(t_n\right) + hx'\left(t_n\right) + \frac{(jh)^2}{2!}x''\left(t_n\right) + \frac{(jh)^3}{3!}x'''\left(t_n\right) + \ldots$$

$$f\left(t_{n+j}, x\left(t_{n+j}\right)\right) = x'\left(t_{n+j}\right) = x'\left(t_n + jh\right) = x'\left(t_n\right) + (jh)\, x''\left(t_n\right) + \frac{(jh)^2}{2!} \times \left(t_n\right)....$$

The truncation error

$$T_{n+k} = \sum_{j=0}^{k} \alpha_j x\left(t_{n+j}\right) = h\sum_{j=0}^{k} \beta_j f\left(t_{n+j}, x\left(t_{n+j}\right)\right)$$

can be written in the form

$$T_{n+k} = c_0 x\left(t_n\right) + c_1 h x^1\left(t_n\right) + c_2 h^2 x''\left(t_n\right) + - - -$$

where

$$c_0 = \alpha_0 + \alpha_1 + \alpha_2 + \ldots + \alpha_k = \sum_{j=0}^{k} \alpha_j$$

$$c_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots + k\alpha_k - (\beta_0 + \beta_1 + \beta_2 + \ldots + \beta_k) = \sum_{j=0}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j$$

$$c_2 = \frac{1}{2!}(\alpha_1 + 2\alpha_2 + 3^2\alpha_3 + \ldots + k^2\alpha_k) - (\beta_1 + 2\beta_2 + \ldots + k\beta_k)$$

$$=\frac{1}{2!}\sum_{j=0}^{k} j^2\alpha_j - \sum_{j=0}^{k} j\beta_j$$

In general,

$$C_q = \frac{1}{q!}\,(\alpha_1 + 2^q\alpha_2 + 3^q\alpha_3 + \ldots + k^q\alpha_k)$$

$$-\frac{1}{(q-1)}\,({}_1\beta + 2^{q-1}\beta_2 + \ldots + k^{q-1}\beta_k)$$

$$= \frac{1}{q!}\sum_{j=0}^{k} j^q\alpha_j - \frac{1}{(1-q)}\sum_{j=0}^{k} j^{q-1}\beta_j$$

This motivates the following definition

Definition 1.2.1[6,7]

We say that a linear multistep method is of order $p \geq 0$ if

$T_{n+k} = c_{p-1}\, h^{p+1}\, x^{p+1}\,(t_n) + 0\,(h^{p+2})$

Or if in the above c0 = c1 =… = cp = 0, cp+1 ≠ 0

Consistency Condition

$$\text{Let } t_{n+k}(x) = \frac{1}{n} T_{n+k}(x) \quad (1.3.1)$$

To show that the approximate solution $\{x_n \mid t_0 \leq t_n \leq b\}$ of (1.2.1) converges to the theoretical solution x (t) of the initial value problem (1.1.1), it is necessary to have

$$\tau(h) = \max_{t_k \leq t_n \leq b} |T_{n+k}(x) \to 0 \text{ as } h \to 0 \quad (1.3.2)$$

This (Aitkinson [1981]) is often called the consistency condition for the method (1.1.2)

We also need to know the condition under which

$\tau(h) = 0(h^m)$ (1.3.3)

for some desired choice $m \geq 1$

Theorem 1.3.1[8,9]

Let $m \geq 1$ be a given integer. In order that Equation (1.3.2) holds for all continuously differentiable functions x (t), that is, that the method (1.12) be consistent, it is necessary and sufficient that

And for Equation (1.3.3) to be valid for all functions x(t) that are m+1 times

$$= \sum_{j=0}^{k}\alpha_j = 1;\ -\sum_{j=0}^{k} j\alpha_j + \sum_{j=0}^{k}\beta_j = 1 \quad (1.3.4)$$

continuously differentiable, it is necessary that

$$\sum_{j0}^{k}(-j)^i\,\alpha_j + i\sum_{j1}^{k}(-j)^{i-1}\,\beta_j = 1,\ i = 2,\ldots,m \quad (1.3.5)$$

Definition 1.4.1[10,11]

A differential equation along with subsidiary conditions on the unknown function and its derivatives, all given at the same value of the independent variable, constitute an initial value problem, where the subsidiary conditions are initial conditions.

MAIN RESULT ON LINEAR MULTISTEP FIXED POINT ITERATIVE METHOD

Analytical Study of the Linear Multistep Methods has Revealed the Following Facts:-

a. That the domain of existence of solution of the linear multistep methods is the complete metric space.
b. That the solution of the linear multistep method converges in the complete metric space.
c. That the initial value problem $x' = f(t,x); x(t_0) = x_0$ solvable by the linear multistep in the complete metric space is a continuous function.
d. That the linear multistep method satisfies the conditions of the Banach contraction mapping principle.
e. That the linear multistep method is exactly the Picard's iterative method with a differential operator instead of the usual integral operator.

Theorem 2.1: Let X be a complete metric space and let R be a region in (t,x) plane containing (t_0, x_0) for $x_0, x \in X$. Suppose, given

$$\dot{x} = f(t,x);\ x(t_0) = x_0 \ldots \quad (2.0)$$

A differential equation where $f(t,x)$ is continuous. If the map f in (2.1) is Lipschitzian and with constant $K<1$, then the initial value problem (2.1) by the linear multistep method has a unique fixed point.

$$x^* = x_{n+k} = \sum_{j=0}^{k-1}\alpha_j x_{n+j} + h\sum_{j=0}^{k}\beta_j f_{n+j}\ldots$$

With a two-step predictor-corrector method.

That is, the predictor $x_{j+1}^{(j)} = \sum_{j=0}^{n}\alpha_j x_j + h\sum_{j=0}^{n}\beta_j f_j$

The corrector: $x_{j+1}^{(j)} = \sum_{j=0}^{n}\pm_j x_{j+1}^{(j)} + h\sum_{j=0}^{n}{}^{2}{}_{j+1} f_{j+1}$

For x_{n+k} satisfying $h < \frac{1}{\ell h|\beta_k|},\ m = \ell h|\beta_k|$

Proof

Let $x_1 = f(x_0)$

$$x_2 = f(x_1) = f(f(x_0)) = f^2(x_0)$$

$$x_3 = f(x_2) = f(f^2(x_0)) = f^3(x_0)$$

$$\vdots$$

$$x_n = f(x_{n-1}) = f(f^{n-1}(x_0)) = f^n(x_0)$$

$$x_{n+k} = f(x_{n+k-1}) = f(f^{n+k-1}(x_0)) = f^{n+k}(x_0)\ldots \quad (2.1)$$

We have constructed a sequence $\{x_n\}_{n=0}$ in (X,ρ). We shall prove that this sequence is Cauchy.

First, we compute

$\rho(x_{n+k}, x_{n+k+1}) = \rho(f(x_{n+k}), f(x_{n+k+1})$ Using (2.1)

$\leq K\rho(x_{n+k-2}, x_{n+k-1})$ Since f is a contraction

$= K\rho(f_{n+k-2}, f_{n+k-1})$ Using (2.1)

$\leq K\left[K\rho(x_{n+k-2}, x_{n+k-1})\right]$ Since f is a contraction

$$= K^2\rho(x_{n+k-2}, x_{n+k-1})$$

$$\vdots$$

$$K^{n+k}\rho(x_0, x_1)$$

That is, $K\rho(x_{n+k}, x_{n+k+1}) \leq K^{n+k}\rho(x_0, x_1)\ldots$ (2.2)

We can now show that $\{x_{n+k}\}_{n=0}$ is Cauchy.

Let m+k>n+k. Then

$$\begin{aligned}\rho(x_{n+k}, x_{m+k}) \leq \rho(x_{n+k}, x_{m+k}) \\ +\rho(x_{n+k-1}, x_{m+k-2}) + \ldots \\ +(x_{n+k-1}, x_{m+k})\end{aligned}$$

$$\leq K^{n+k}\rho(x_0, x_1) + K^{n+k-1}\rho(x_0, x_1) + \ldots$$

$$+K^{n+k-1}\rho(x_0, x_1)$$

Using Equation (2.2)

Since the series on the right hand side is a geometric progression with common ratio <1, it sum to infinity is $\frac{1}{1-k}$. Hence, we have from above that

$$\rho(x_n, x_m) \leq k^{n-k}\rho(x_0, x_1)\left(\frac{1}{1-k}\right) \to 0 \text{ as}$$

$$n - k \to \infty \text{ since } k < 1$$

Hence, the sequence $\{x_{n+k}\}_{n=0}$ is a Cauchy sequence in X and since X is complete, $\{x_{n+k}\}_{n=0}^{\infty}$. Converges to a point in *X*.

Let $x_{n+k} \to x^*$ as $h \to \infty\ldots$ (2.3)

Since f is a contraction and hence is continuous, it follows from Equation (2.3) that $f(x_{n+k}) \to f(x^*)$ as $n\to\infty$. But $f(x_{n+k})=x_{n+k+1}$ from (2.2). So

$$x_{n+k+1} = f(x_{n+k}) = f(x^*)\ldots \quad (2.4)$$

limits are unique in a metric space, so from Equations (2.3) and (2.4), we obtain that

$$f(x^*) = x^*\ldots \quad (2.5)$$

Hence, f has a unique fixed point in X. We shall now prove that this fixed point is unique. Suppose for contradiction, there exists y*∈ X such that

$$y^* = x^* \text{ and } f(y^*) = y^*\ldots \quad (2.6.)$$

Then, from Equations (2.5) and (2.6)

$$\rho(x^*, y^*) = \rho(f(x^*), f(y^*)) \leq k\rho(x^*, y^*)$$

So that

$$(k-1)\ \rho(x^*, y^*) \geq 0 \text{ and } (k-1)\ \rho(x^*, y^*) \geq 0$$

We can divide by it to get k−1≥0 i.e k≥1 which is a contradiction.

Hence x*=y* and the fixed point is unique.

Therefore

$$x_{n+k} = \sum_{j=0}^{k-1} \pm_j x_{n+j} + h\sum_{j=0}^{k}{}^{2}{}_{j} f_{n+j}; a \leq t_n \leq n$$

Is the linear multistep fixed point iterative formula for the initial value problem

$$\dot{x} = f(t, x);\ \ x(t_0) = x_0$$

Of the ordinary differential type.

Finally, to be sufficiently sure, we also show that

$$x^* = x_{n+k} = \sum_{j=0}^{k-1}\alpha_j x_{n+j} + h\sum_{j=0}^{k}\beta_j f_{n+j}$$

Satisfies the Lipschitz condition.

$$|x^* - y^*| = |x_{n+k} - y_{n+k}|$$

$$= \left|\left(\sum_{j=0}^{k-1}\alpha_j x_{n+j} + h\sum_{j=0}^{k}\beta_j f_{n+j}\right) - \left(\sum_{j=0}^{k-1}\alpha_j y_{n+j} + h\sum_{j=0}^{k}\beta_j f^*_{n+j}\right)\right|$$

$$\le \sum_{j=0}^{k-1} \pm_j \left|x_{n+j} - y_{n+j}\right| + h\sum_{j=0}^{k} {}^2{}_j \left|f_{n+j} - f^*_{n+j}\right|$$

$$= k_1 \sum_{j=0}^{k-1}\left|x_{n+j}-y_{n+j}\right| + k_2\sum_{j=0}^{k}\left|f_{n+j}-f^*_{n+j}\right|$$

$$= \left(k_1+k_2\right)\left|\sum_{j=0}^{k-1} z_{n+j} + \sum_{j=0}^{k} f_{m+k}\right|$$

$$= K\sum_{j=0}^{k}\left|z_{n+j}+f_{m+j}\right|$$

Hence $x^*=x_{n+k}$ is Lipschitzian and hence is a continuous map with the above fixed point.
Respectively, iterative methods for the respective linear multistep methods are as follows:

The Explicit Methods Are

i. Euler:

$$x_{n+1} = x_n + hf_n$$

ii. The midpoints method:

$$x_{n+2} = x_n + 2hf_{n+1}$$

iii. Milne's method:

$$x_{n+1} = x_{n-3} + \frac{4h}{3}\left[2f_{n-2} + f_{n-1} + 2f_n\right]$$

iv. Adam's method:

$$x_{n+1} = x_n + \frac{h}{24}\left[55f_n - 59f_{n-1} + 35f_{n-2} - 9f_{n-3}\right]$$

v. The Generalized predictor method:

$$x^{(j)}_{j+1} = \sum_{j=0}^{n} \pm_j x_j + h\sum_{j=1}^{n} {}^2{}_j f_j$$

The Implicit Methods Are

i. Trapezoidal method:

$$x^{(j+1)}_{n+1} = x_n + \frac{h^2}{6}\left[f_n + f^{(j)}_{n+1}\right]$$

ii. Simpson's method:

$$x^{(j+1)}_{n+2} = x_n + \frac{h}{3}\left[f^{(j)}_{n+1} + 4f_{n+1} + f_n\right]$$

iii. Simpson's method:

$$x^{(j+1)}_{n+2} = x_{n-2} + \frac{h}{3}\left[f_{n-1} + 4f_n + f^{(j)}_{n+1}\right]$$

iv. Adams Moulton's method

$$x^{(j+1)}_{n+2} = x_n + \frac{h}{3}\left[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right]$$

v. Milne's corrector method:

$$x^{(j+1)}_{n+1} = x_{n-2} + \frac{h}{3}\left[f_{n-1} + 4f_n + f^{(j)}_{n+1}\right]; n = 1$$

vi. The Generalized corrector method

$$x^{(j)}_{j+1} = \sum_{j=0}^{n} \pm_j x^{(j)}_{j+1} + h\sum_{j=1}^{n} {}^2{}_{j+1} f_{j+1}$$

C: The generalized two step (corrector predictor) method

$$x^{(j)}_{j+1} = \sum_{j=0}^{n}\alpha_j x_j + h\sum_{j=1}^{n}\beta_j \ldots \quad (C1)$$

$$x^{(j)}_{j+1} = \sum_{j=0}^{n}\alpha_j x^{(j)}_{j+1} + h\sum_{j=1}^{n}\beta_{j+1} f_{j+1} \ldots \quad (C2)$$

Here $x^{(j+1)}_{j+1} \in X$ are the corrector points to be determined for all j≥0 while $x^{(j+1)}_{j+1} \in X$ are predetermined before $x^{(j+1)}_{j+1} \in X$. While the iterations are alternatively implement one after the other starting first with the predictor.

Note: The generalized compact form of C1 and C2 is as follows

$$x_{n+k-1} = \sum_{j=0}^{k-1}\alpha_j x_{n+j} + h\sum_{j=0}^{k}\beta_j f_{n+j}$$

CONSISTENCY AND STABILITY THEORY

In this section, the theory of consistency and stability (leading to convergence) is presented for the linear multistep method.
Given the linear multistep method

$$x_{n+k} = \sum_{j=0}^{k-1}\alpha_j x_{n+j} + h\sum_{j=0}^{k}\beta_j F_{n+j}; a \le t_n \le n \ldots \quad (2.7)$$

Theorem 2.2 Consistency[12-14]

Let $x_{n+j}=x(t_{n+j})$;j=0,1,2,…,k−1 denote its numerical solution

$$T_{n+k} = \sum_{j=0}^{k-1}\alpha_j x\left(t_{n+j}\right) - h\sum_{j=0}^{k}\beta_j f\left(t_{n+j},\left(x_{n+j}\right)\right)$$

The local truncation error and

$$\tau_{n+k} = \frac{1}{h}T_{n+k}(x)$$

Then, the linear multistep method (3.1) is said to be consistent if

$$\tau(h) = \max\left|T_{n+k}(x)\right| \to 0 \text{ as } h \to 0 \text{ and}$$

$$\sum(h) = 0(h^m)$$

For some m≥1 or equivalently, Equation (1.3.2) is said to be consistent if

$$\sum_{j=0}^{k}\alpha_j; \sum_{j=0}^{k} j\alpha_j + \sum_{j=1}\beta_j = 1\ldots \quad (2.8)$$

Proof:

If the numerical solution of a given linear multistep method is

$$x_{n+j} = x(t_{n+j}), j = 0,1,2,\ldots,k-1\ldots \quad (2.9)$$

And the local truncation error is

$$T_{n+k} = \sum_{j=0}^{k-1}\alpha_j x\left(t_{n+j}\right) - h\sum_{j=0}^{k}\beta_j f\left(t_{n+j}, x_{n+j}\right)\ldots \quad (2.10)$$

With $\tau_{n+k}\left(x\right) = \frac{1}{h}T_{n+k}\left(x\right),\ldots$ (2.11)

We want to prove that the linear multistep method

$$x_{n+k} = \sum_{j=0}^{k-1}\alpha_j x_{n+j} + h\sum_{j=0}^{k}\beta_j f_{n+j}, a \le t_{n+j} \le n \quad (2.12)$$

Is consistent if

$$\tau\left(h\right) = \max\left|T_{n+k}(x)\right| \to 0 \text{ as } h \to 0\ldots \quad (2.13)$$

And

$\tau\left(h\right) = 0(h^m)$ For some $m \ge 1\ldots$ (2.14)

If $\bar{x}_{n+j}$ denotes the numerical solution with the above exact values (1.2), then Equation (2.14) yields

$$x_{n+k} + \sum_{j=0}^{k-1}\pm_j x_{n+j} = h^2{}_k f\left(t_{n+k}, \bar{x}_{n+k}\right)$$
$$+h\sum_{k=0}^{k-1}{}^2 f\left(t_{n+j}, x_{n+j}\right)\ldots \quad (2.15)$$

Applying localizing assumption on Equation (2.15) means that no previous truncation error has been made and that

$$x_{n+j} = x\left(t_{n+j}\right),\ j = 0,1,\ldots,k-1$$

So that we have

$$\bar{x}_{n+k} + \sum_{j=0}^{k}\alpha_j \bar{x}(t_{n+j}) = h\beta_k f\left(t_{n+k}, \bar{x}_{n+k}\right)$$
$$+h\sum_{k=0}^{k-1}\beta_j f\left(t_{n+j}, \bar{x}_{n+j}\right)\ldots (2.16)$$

Using the local truncation error earlier defined in Equation (2.2) we now have

$$x(t_{n+k}) + \sum_{j=0}^{k-1}\alpha_j x(t_{n+j})) = T_{n+k} + h\beta_k f\left(t_{n+k})\right), x\left(t_{n+k}\right)$$
$$+h\sum_{k=0}^{k-1}\beta_j f\left(t_{n+j}, x(t_{n+j})\right)\ldots$$
$$(2.17)$$

Subtracting Equation (2.16) from Equation (2.17) we have

$$x(t_{n+k}) - \bar{x}_{n+k} = T_{n+k} + h\beta_k[f\left(t_{n+k}, x(t_{n+k})\right)$$
$$-f\left(t_{n+k}, \bar{x}_{n+k}\right)\ldots \quad (2.18)$$

If we apply mean value theorem on (3.1.10),
We have

$$f\left(t_{n+k}, x(t_{n+k})\right) - f\left(t_{n+k}, \bar{x}_{n+k}\right) =$$
$$\left(x\left(t_{n+k}\right) - \bar{x}_{n+k}\right)\frac{\partial f}{\partial x}\Big|(t_{n+k,}\tau_{n+k})$$

Where η_{n+k} lies between $\bar{x}_{n+k}$ and $x(t_{n+k})$

Therefore

$$\left[1 - h\beta_k\frac{\partial f}{\partial x}\Big|\left(t_{n+k}, \eta_{n+k}\right)\right]\left(x\left(t_{n+k}\right), \bar{x}_{n+k}\right) = T_{n+k}\ldots$$
$$(2.19)$$

Let e_{n+k} represents the error at (n+k) point, so that if the method is explicit β_k=0 and then T_{n+k}=e_{n+k} but if the method is implicit β_k≠0 and

$h\beta_k\left(\frac{\partial f}{\partial x}\right)\Big|\left(t_{n+k}, n_{n+k}\right)$ is small then $T_{n+k} \approx e_{n+k}$

Again let

$$\tau_{(n+k)}\left(x\right) = \frac{1}{h}T_{n+k}\left(x\right)\ldots \quad (2.20)$$

For us to show that the approximate solution $\{x_n \left| t_0 \le t_n \le b\right\}$ of (3.1.4) converges to the theoretical solution x(t) of the initial value problem $\dot{x} = f\left(t,x\right); x\left(t_0\right) = x_0$

We need to necessarily satisfy the consistency condition

$$\tau\left(h\right) = \max_{t_0 \le t_n \le b}\left|T_{n+k}(x)\right| \to 0 \text{ as } h \to 0\ldots \quad (2.21)$$

Plus the condition that

$$\tau\left(h\right) = 0(h^m)\text{, for some } m \ge 1\ldots \quad (2.22)$$

By this, we[15-17] show the only necessary and sufficient condition for the linear multistep (2.14) to be consistent is that

$$\sum_{j=0}^{k}\alpha_j = 1 \text{ and } -\sum_{j=0}^{k} j\alpha_j + \sum_{j=1}^{k}\beta_j = 1\ldots \quad (2.23)$$

And for Equation (2.23) above to be valid for all functions, x(t) is for x(t) that are m+1 times continuously differentiable to necessarily satisfy

$$\sum_{j=0}^{k}(-j)\alpha_j + \sum_{j=1}^{k}(-j)^{i-1}\beta_j = 1, i = 2,\ldots m\ldots \quad (2.24)$$

Hence, we know that

$$T_{n+k}(\alpha x + \beta w) = \alpha T_{n+k}(x) + \beta T_{n+k}(w)\ldots \quad (2.25)$$

For all constants α,β and all differentiable functions x,w. We now examine the consequence of Equations (2.19) and (2.20) by expanding x(t) about t_n using Taylor's theorem and we have

$$x(t) = \sum_{j=o}^{k}\frac{1}{j}(t - t_0)\ x^1(t_n) + R_{m+1}(t) \quad (2.26)$$

Assuming x(t) is m+1 times continuously differentiable. Substituting into the truncation error

$$T_{n+k}(x) = x(t_{n+k}) - \sum_{j=0}^{k}\alpha_j x(t_{n+j}) + h\sum_{j=1}^{k}\beta_j F(t_{n+j})\ldots \quad (2.27)$$

And also using Equation (2.23)

$$T_{n+k}(x) = \sum_{j=0}^{m-1}\frac{1}{j}x^{(j)}(t_n)\ T_{n+k}((t - t_n)^j) + T_{n+k}(R_{m+1})\ldots \quad (2.28)$$

It becomes necessary[18-20]. To calculate

$$T_{n+k}(t - t_n)^j \text{ for } j = 0$$

$$T_{n+k}(1) = c_0 \equiv 1 - \sum_{j=0}^{k}\pm_j\ldots \quad (2.29)$$

For j≥1 we have

$$T_{n+k}(t - t_n)^j = T_{n+k}(t - t_n)^j$$

$$= \left(\sum_{j=0}^{k}\alpha_j(t_{n+k} - t_n)^j + h\sum_{j=0}^{k}\beta_j^i(t_{n+j} - t_n)^{i=1}\right) = c_1 h^1\ldots \quad (2.30)$$

$$C_j = 1 - \left(\sum_{j=0}^{k}(-j)^i\alpha_j + i\sum_{j=1}^{k}(-j)^{i=1}\beta_j\right), i \geq 1$$

This gives

$$T_{n+k}(x) = \sum_{j=1}^{m}\frac{c}{j}h^j x^{(j)}(t_n) + T_{n+k}(R_{m+1})\ldots \quad (2.31)$$

And if we write the remainder R_{m+1} (t) as

$$R_{m+1}(t) = \frac{1}{(m+1)!}(t - t_n)^{m+1}x^{m+1}(t_n) + \ldots$$

Then

$$T_{n+k}R_{m+1}(t) = \frac{C_{m+1}}{(m+1)!}h^{m+1}x^{m+1}(t_n) + 0(h^{m+2})\ldots \quad (2.32)$$

To obtain the consistency condition (2.20), we need τ(h)-0(h) and this requires T_{n+k} (x)=0(h^2). Using Equation (2.19) with m=1, we must have C_0, C_1=0 which gives the set of Equations (2.22) which are referred to as consistency conditions in some texts. Finally to obtain (3.1.14) for some m≥1, we must have T_{n+k} (x)=0(h^{m+1}). from Equations (2.30) and (2.31), this will be true if and only if C_i=0,i=0,1,2…m.

This proves the conditions (2.21) and completes the proof.

Theorem 2.3 Stability:[21-23] Assume the consistency condition of Equation (2.10), then the linear multistep method (1.2) is stable if and only if the following root conditions (2.11)-(2.12) are satisfied

The root $|r_j| < 1 = j = 0,1,\ldots,k\ldots \quad (2.33)$

$$|r_j| = 1 \Rightarrow \rho^1(r_j) \neq 0\ldots \quad (2.34)$$

Where

$$\rho(r) = r^{k-1} - \sum_{j=0}^{k}\alpha_j r^j$$

Proof:

Given the linear multistep

$$x_{n+k} + \sum_{j=0}^{k-1}\alpha_j x_{n+j} + h\sum_{k=0}^{k-1}\beta_j f_{n+j}; a \leq t_{n+j} \leq n\ldots \quad (2.35)$$

With the associated characteristic polynomial

$$P(r) = r^{k+1} - \sum_{j=0}^{k}\alpha_j r^j\ldots \quad (2.36)$$

Such that P(1)=0 by the consistency condition. Let $r_0,\ldots,r_n$ denote the respective roots of P(r), repeated according to their multiplying and let r_0=1.

The linear multistep method 2.8. Satisfies the root condition if

$$\left|r_j\right| \le 1, j = 0, 1, \ldots, k \ldots \quad (2.37)$$

$$\left|r_j\right| = 1 \Rightarrow P^1\left(r_j\right) \ne 0 \ldots \quad (2.38)$$

Let Equation (2.8) be stable, we now prove that the root conditions (2.37) and (2.38) are satisfied. By contradiction let

$|r_j (0)|>1$ for some j. This is to say we consider the initial value problem $x^1 \equiv 0; x(0)=0$ with solution $x(t)=0$. So that Equation (2.8) becomes

$$x_{n+k} = \sum_{j=0}^{k-1} \alpha_j x_{n+j}; n \ge k \ldots \quad (2.39)$$

If we take $x_0=x_1=\ldots=x_k=0$, then the numerical solution clearly becomes $x_n=0$ for all $n\ge 0$.

For perturbed initial values, let

$$z_0 = \in, z_1 = \in r_1\left(0\right), \ldots, z_n = \in r_1(0)^p \ldots (2.40)$$

And for these initial values

$$\max_{0\le n-k}\left|x_n - z_n\right| \le \in \left|r_1(0)\right|^p$$

Which is a uniform bound for all small values of h, since the right side is independent of h, as $\in \rightarrow 0$, the bound also tend to zero.

The solution (2.12)[24-26] with the initial condition (2.13) is simply $z_n = \in r_j\left(0\right)^n; n \ge 0$. For the derivation from $\{x_n\}$

$$\max_{0\le n-k}\left|x_n - z_n\right| = N(h) \rightarrow \infty$$

And the bound that the method is unstable when $|r_j (0)| > 0$. Hence, if the method is stable, the root condition $|r_j (0)| \le 1$. Must be satisfied.

Conversely, assume the root condition is satisfied, we now prove for stability restricted to the exponential equation.

$$x^1 = \lambda x; \quad x\left(0\right) = 1.. \quad (2.41)$$

This[27-29] involves solution of non-homogenous linear difference equations which we simplify by assuming the roots $r_j (0)$; $j=0,1,\ldots,k$ to be distinct. The same will be true of $r_j (h\lambda)$ provided the values of h is kept sufficiently small, say $0\le h\le h_0$. Assume $\{x_n\}$ and $\{z_n\}$ to be two solutions of

$$\left(1 - h\lambda\beta_{k+1}\right) x_{n+k+1} - \sum_{j=0}^{k-1}(\alpha_j + h\lambda\beta_j)x_{n+j} = 0; n \ge 1 \ldots \quad (2.42)$$

On Equation (2.10) on $[x_0, b]$ and assume that

$$\max_{0\le n-k}\left|x_n - z_n\right| \le \in, 0 \le h \le h_0$$

Introduce the error $e_n=x_n-z_n$ and subtracting using (2.3.8) for each solution

$$\left(1 - h\lambda\beta_k\right) e_{n+k} - \left(\alpha_j + h\lambda\beta_j\right)e_j = 0; x_{k+1} \le x_{n+k} \le b \ldots \quad (2.43)$$

The general equation becomes

$$e_n = \sum_{j=1}^{k}\gamma_j \left|r_j(h\lambda)\right|^n; n \ge 0 \quad (2.44)$$

The coefficient $\gamma_0,\ldots,\gamma_k$ must be chosen so that the solution (2.17) will then agree with the given initial perturbations $e_0,\ldots,e_k$ and will satisfy the difference Equation (2.16). Using the bound $z_0=\in$, $z_1=\in r_1(0),\ldots,z_n=\in r_j(0)^p$ and the theory of linear system of equations, we have

$$\max_{0\le n-k}\left|\gamma_n\right| \le c\varepsilon; 0 \le h \le h_0 \ldots \quad (2.45)$$

for some constants $c_j>0$.

To bound the solution e_n on $[x_0,b]$, we must bound each term $[r_j (h\lambda)]^n$ to do so, consider the expansion

$$\left(U\right) = r_j\left(0\right) + Ur_j\left(\xi\right)\ldots \quad (2.46)$$

For some ξ between 0 and U. To compute $r_j^1(u)$, differentiate the identity

$$p\left(r_j\left(u\right)\right) - u\sigma\left(r_j\left(u\right)\right) = 0$$

$$p^1\left(r_j\left(cu\right)\right) - r_j^1\left(u\right) - \left[\sigma_j\left(u\right) + u\sigma^1\left(r_j\left(u\right)\right)\left(r^1\left(u\right)\right)\right] = 0$$

$$r_j^1\left(u\right)[p^1\left(r_j\left(u\right)\right) - u\sigma^1\left(r_j\left(u\right)\right) = \sigma(r_j\left(u\right))$$

$$r_j^1\left(u\right) = \frac{\sigma\left(r\left(u\right)\right)}{p\left(r_j\left(u\right)\right) - ur^1\left(r\left(u\right)\right)}\ldots \quad (2.47)$$

By assumption that $r_j (0)$ is a simple root of $p(r)=0; 0\le j\le k$, it follows that $p^1 (r_j (0))=0$ and by continuity, $p^1 (r_j (u))\ne 0$ for all sufficiently small values of u, the denominator in (2.20) is non-zero and we can bound $r_j (u)|r_j (u)|=c_2$ for all $|u|\le u_0$ For some $U_0\ge 0$.

Using this with (3.3.12) and the root condition (3.2.4), we have

$$\left|[r_j\left(h\lambda\right)]^n\right| \le \left|r_j\left(0\right)\right| + c_2\left|\left(h\lambda\right)\right| \le 1 + c_2\left|\left(h\lambda\right)\right|$$

$$\left|[r_j\left(h\lambda\right)]^n\right| \le [1 + c_2\left|\left(h\lambda\right)\right|]^n \le e^{C_2 h|h\lambda|} \le e^{c_2}(bx_n) \ |\lambda|$$

for all $0 \le h \le h_0$.

Combine this with Equations (2.18) and (2.19) to get $\mathrm{Max}\left|e_n\right| \le c_2 \le |\varepsilon| e^{c_2}(bx_n)\ \ |\lambda|$ for an approximate constant c_0. This concludes the proof.

Theorem 2.4 convergence[29-31/

The linear multistep method (1.2) is said to be convergent if and only if it is consistent and stable.

Proof

By this, we want proof that if the consistency condition is assumed, the linear multistep method

$$x_{n+k} = \sum_{j=0}^{k-1} \alpha_j x_{n+j} + h \sum_{j=0}^{k} \beta_j f_{n+j}; a \le t_{n+j} \le n \qquad (2.48)$$

Is convergent if and only if the root conditions (2.10) and (2.11) are satisfied.

We assume first the root conditions are satisfied and then show the linear multistep (2.8)

Is convergent. To start, we use the problem x=0, x(0) =0 with the solution x (t) = 0. Then, the multistep method (2.8) becomes

$$x_{n+k} = \sum_{j=0}^{k-1} \alpha_j x_{n+j}, n \ge k\ldots \qquad (2.49)$$

With $x_0,\ldots,x_k$

Satisfying n (h) = $\max|x_n| \to 0$ as $h \to 0\ldots$ (2.50)

Suppose[30,31] that the not condition is violated, we will show that Equation (2.10) is not convergent to x(t)=0. Assume that some $\left|r_j(0)\right| > 1$ then a satisfactory solution of Equation (2.11) is

$$x_n = h[r_j(0)]^n; t_0 \le t_n \le b\ldots \qquad (2.51)$$

Condition (2.10) is satisfied since $n(h) = \left|h(r_j(0))\right| \to 0$ as $h \to 0$.

However, the solution (2.11) does not converge. First

$$\mathrm{Max}\left|x(t_n) - x_n\right| = h\left|h[r_j(0)\right|^{N(h)}\ 0 \le t_n \le b$$

Consider those values of $h = \frac{b}{N(h)}$. Then, L' Hospital's rule can be used to show that

$$\mathrm{Lim}\frac{b}{N}\left|r(0)\right|^N = \infty$$

Showing that Equation (2.11) does not converge.

Conversely assume the root condition is satisfied as with theorem 2.2; it is rather difficult to give a general proof of converge for an arbitrary differential equation. The present proof is restricted to the exponential Equation (2.14) and again we assume that the roots $r_j = 0$ are distinct. To simplify the proof, we will show that the term $\gamma_0\ [r_0\ (\lambda\lambda)]^n$ in the solution

$$x_n = \sum_{j=0}^{k} {}^{3}{}_{j} \left|r_j(h\lambda)\right|^n$$

Will converge to the solution $e^{\lambda t}$ on [0,b]. The remaining terms

$\gamma_j\left|r_j(h\lambda)\right|^n, j = 1,2,\ldots,k$ are parasitic solution to converge to zero as h→0. Expand r_0 (hλ) using Taylor's theorem,

$$r_0(h\lambda) = r_0(0) + h\lambda r_0^2(0) + 0(h^2)$$

From Equation (2.19) $r_0^2(0) = \frac{\sigma(1)}{\rho^1(1)}$ and using this consistency condition (2.11), this leads to $r_0^2(0) = 1$. Then

$$r_0(h\lambda) = 1 + h\lambda + 0(h^2) = e^{h\lambda} + 0(h^2)$$

$$[r_0(h\lambda)]^n = e^{hn\lambda}[1 + 0(h^2)]^n = e^{\lambda tn}[1 + 0(h^2)]$$

Thus

$$\max_{0 \le t_n \le b}\left|[r_0(h\lambda)] = e^{\lambda t_n}\right| \to 0 \text{ as } h \to 0\ldots \qquad (2.52)$$

We[30,31] must now show that the coefficient $\gamma_0 \to 1$ as $h \to 1$. The coefficient $\gamma_0,\ldots,\gamma_k$ satisfy the linear system

$$\gamma_0 + \gamma_1 + \ldots + \gamma_k = x_0$$

$$\gamma_0\left[r_0(h\lambda)\right] + \ldots + \gamma_k[r_k(h\lambda)] = x_1$$

$$\gamma_0\left[r_0(h\lambda)\right]^k + \ldots + \gamma_k[r_k(h\lambda)]^k = x_2\ldots.. \qquad (2.53)$$

The initial values $x_0,\ldots,x_k$ are assumed to satisfy

$$r_j(h)\max_{0 \le n-k}\left|e^{\lambda t_n} - x_n\right| \to 0 \text{ as } h \to 0$$

But this implies

$$\lim x_n = 1, 0 \le n \le p\ldots \qquad (2.54)$$

The coefficient γ_0 can be obtained using Cramer's rule to solve (2.53) then

$$\gamma_0 = \frac{\begin{vmatrix} x_0 & 1 & \cdots & 1 \\ x_1 & r_1 & \cdots & r_k \\ \vdots & & & \\ x_k & r_k & \cdots & r_k^k \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_0 & r_1 & \cdots & r_k \\ \vdots & & & \\ r_k^k & r_1^k & \cdots & r_k^k \end{vmatrix}}$$

The denominator converges to the Vandermonde determinant for $r_0(0)=1$, $r_1(0),\ldots, r_k(0)$; and this is non-zero since the roots are distinct. Using Equation (2.13), the numerator converges to the same quantity as $h\rightarrow 0$. Therefore $\gamma\rightarrow 1$ as $h\rightarrow 0$, using this, along with Equation (2.10), the solution $\{x_n\}$ converges to $x(t)=e^{\lambda t}$ on [0,b]. This completes the proof.

ILLUSTRATIVE EXAMPLES ON STABILITY AND CONVERGENCE

Example 3.1

Illustrate the effect of stability using the linear multistep method

$$x_{n+2} = (1+a)\ x_{n+1} + ax_n$$

$$= h[(3-a)\ f_{n+1} - \frac{1}{2}(1+a)f_n] \text{ with}$$

i. $a = 0$
ii. $a = -1$
iii. $a = -5$ to compute numerically solutions to the initial value problem; $x = 4tx^{½}$; $x(0) = 1$ in the interval $0\leq t\leq 2$.

Solution

$p(r) = r^2 - (1+a)\,r+a = (r-1)\,(r-a)$

For a = 0 and −1, obviously, the method is stable because the stability condition (2.2.9), (2.2.10) is satisfied. For a = −5, we have $r^2+4r-5 = (r-1)(r+5) \Rightarrow r=1$ and −5 but $|r_2| = 5>1$. This violates the stability condition and so far a = −5 the linear multistep method has order 3 for a = −5 and order 2 otherwise. The theoretical solution is $x(t) =(1+t^2)^2$. Let $x_0=1$ and we also choose the necessary starting value x_1 to coincide with the theoretical solution; that is $x_1=(1+t^2)^2$. Then, we generate the solutions provided in the table below using h=0.1.

$a = 0 \Rightarrow x_{n+2} - x_{n+1} + \frac{1}{2}\ [3f_{n+1} - f_n]$

$a = -1 \Rightarrow x_{n+2} - x_n + 12h\ [f_{n+1}]$

$a = -5 \Rightarrow x_{n+2} - 5x_n - 4x_{n+1} + 2h\ [2x_{n+1} - x_n]$

n	t	Theoretical solution	a=0; xn	a=−1; xn	a=−5; xn
0	0.0	1,000,000	1,000,000	1,000,00	1,000,000
1	0.1	1,020,100	1,020,100	1,020,100	1,020,100
2	0.2	1,081,600	1,080,800	1,081,800	1,081,200
3	0.3	1,188,100	1,185,248	1,186,438	1,189,238
4	0.4	1,345,600	1,339,630	1,342,217	1,338,866
5	0.5	1,562,500	1,552,090	1,557,171	1,592,993
6	0.6	1,849,600	1,833,245	1,841,364	1,702,339
7	0.7	2,220,100	2.196,092	2,208,516	9.942,623
8	0.8	2.689,600	2.656,023	2.673,584	−27.100945
9	0.9	2.276,100	3.230,824	3.254,987	
10	1.0	4.000,000	3.940,690	3.972,578	
11	1.1	4.884,100	4.808,219	4.849,493	
12	1.2	5.953,600	5.858,421	5.910,475	
13	1.3	7.236,600	7.118,713	7.183,394	
14	1.4	8.761,600	8.618,925	8.697,868	
15	1.5	10.562,500	10.389,007	10.486,514	
16	1.6	12.673,600	12.467,957	12.583,814	
17	1.7	15.132,100	14.890,757	15.027,145	
18	1.8	17.977,600	17.696,868	17.855,836	
19	1.9	21.252,100	20.928,164	21.112,033	
20	2.0	25.000,000	24.628,922	24.839,906	

Example 3.2

Illustrate the effect of inconsistency using the linear multistep method $x_{n+2} - x_{n+1} = \frac{h}{3}(3f_{n+1} - 2f_n)$ to compute a numerical solution for the initial value problem of example (3.1) in the interval $0\leq t\leq 1$

Solution

Note that $\alpha_2 = 1$, $\alpha_1 = 1$, $\alpha_0 = 0$

$\beta_2 = 0$, $\beta_1 = 1$, $\beta_0 = -\frac{2}{3}$

$$\sum_{j0}^{k} j\alpha_j = 0 \neq 1$$

$$\sum_{j0}^{k} j\alpha_j + \sum_{j0}^{k} j\beta_j = \frac{1}{3} \neq 1$$

Hence, the linear multistep method does not satisfy the consistency criteria and so is inconsistent.

n	t	Theoretical solution	h=0.1	t	h=0.05	t	h=0.025
0	0.0	1,000,000	1,000,000	0.00	1,000,000	0.00	1,000,000
1	0.1	1.020,100	1,020,100	0.05	1.005,006	0.025	1.100,250
2	0.2	1.081,600	1,060,500	0.10	1.015,031	0.050	1.103,752
3	0.3	1.188,100	1,115,951	0.15	1.028,498	0.075	1.007,074
4	0.4	1.345,600	1,187,948	0.20	1.045,489	0.100	1.011,281
5	0.5	1.562,500	1.277,821	0.25	1.070,838	0.125	1.016,319
6	0.6	1.849,600	1.387,633	0.30	1.095,312	0.150	1.022,216
7	0.7	2.220,100	1.519,629	0.35	1.123,613	1.175	1.028,981
8	0.8	2.689,600	1.676,318	0.40	1.155,951	0.200	1.036,622
9	0.9	3.276,100	1.860,521	0.45	1.192,496	0.225	1.045,151
10	1.0	4.000,000	2.075,359	0.50	1.233,436	0.250	1.054,578
11				0.55	1.278,975	0.275	1.064,916
12				0.60	1.329,336	0.300	1.076,176
13				0.65	1.384,755	0.325	1.088,382
14				0.70	1.445,499	0.350	1.101,540
15				0.75	1.511,834	0.375	1.115,670
16				0.80	1.584,055	0.400	1.130,790
17				0.85	1.662,473	0.425	1.146,723
18				0.90	1.747,416	0.450	1.164,078
19				0.95	1.839,229	0.475	1.187,287
20				1.0	1.938,276	0.500	1.201,567
21						0.525	1.221,143
22						0.550	1.243,438
23						0.575	1.266,079
24						0.600	1.289,891
25						0.625	1.314,903
26						0.650	1.341,142
27						0.675	1.368,638
28						0.700	1.397,422
29						0.725	1.427,526
30						0.750	1.448,935
31						0.775	1.481,466
32						0.800	1.515,609
33						0.825	1.537,555
34						0.850	1.574,195
35						0.875	1.612,643
36						0.900	1.652,661
37						0.925	1.694,284
38						0.950	1.737,552
39						0.975	1.782,510
40						1.000	1.829,199

Now $p(r) = r^2 - r = r(r-1)$, so that the roots are $r = 0$ and $+1$ hence the linear multistep is stable. Therefore, example (4.5.2) is an example of an inconsistent stable linear multistep method.

$$x_{n+2} = x_{n+1} + 6(\frac{h}{3}3f_{n+1} - 2f_n)$$

Observe that as $h \rightarrow 0$, the numerical solution moves away from the theoretical solution.

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