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RESEARCH ARTICLE

On the Review of Linear Multistep Methods as Fixed Point Iterative Methods in the Solution of a Coupled System of Initial Value Differential Problem

Eziokwu C. Emmanuel

Departments of Mathematics, Micheal Okpara University of Agriculture, Umudike, Abia, Nigeria

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ABSTRACT

The linear multi-step method, $x_{n+k}=\sum_{j=0}^{k-1}\alpha_j x_{n+j}+h\sum_{j=0}^{k}\beta_j\beta_j f_{n+j}; k\geq 1,$ a numerical method for solving the initial value problem $x^{'}=f(t,x), x(t_0)=x_0$ is hereby analytically reviewed and observations show that the linear multistep method is a typical Picard's fixed point iterative formula with a differential operator endowed with some numerical reformulations instead of the usual integral operator as in the traditional Picard's method. The study also shows that any given linear multistep iterative method will be convergent to a fixed point if and only if it is consistent and stable. This was adequately illustrated with an example as is contained in section three of this work.

Key words: Linear multistep, Fixed point iterative method, Consistence, Stability convergence

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INTRODUCTION

Consider the differential equation

$$\dot{x}=f(t,x);x(t_0)=x_0\ldots \tag{1.1}$$

A computational method for determining the sequence $\{x_n\}$ that takes the form of a linear relationship between x_{n+j}, f_{n+j};=0,1,…k is called the linear multistep method of step number k or k– step method;

$$f_{n+j}=f\left(t_{n+j},x_{n+j}\right)=f\left(t_{n+j},x\left(t_{n+j}\right)\right).$$

The general linear multistep method[1-4] may be given thus –

$$x_{n+k}=\sum_{j=0}^{k-1}\alpha_j x_{n+j}+h\sum_{j=0}^{k}\beta_j\beta_j f_{n+j}\ldots \tag{1.2}$$

Where α_j and β_j are constants, $\alpha_k\neq 0$ and not both α_0 and β_0 are zeros. We may without loss of generality, and for the avoidance of arbitrariness set α_k=1 throughout.

Suppose in (1.2), β_k=0 then it is called an explicit method since it yields the current value x_{n+k} directly in terms ofx_{n+j}, f_{n+j}; j = 0,1,…,k–1 which by the stage of computation have already been calculated.[5-8] If, however, $B_k\neq 0$, then (1.2) is called an implicit method which requires the solution at each stage of the computation of the equation

$$x_{n+k}=h\beta_k f+(t_{n+k},x_{n+k}+g\ldots \tag{1.3}$$

Were g is a known function of the previously calculated values, x_{n+j}, f_{n+j}; j=0,1,…,k–1

If the lipschittz condition is applied onfand $m=lh|B_k|$, we observe that the unique solution exists as seen in the theorem (3.1) and that the unique solution for x_{n+k} exists as well where the computational iteration converges to x_{n+k} if

$$\ell h|B_k|<|\text{i.e.}h<\frac{1}{\ell h|^{2}{}_{k}|}$$

We, therefore, see that;[9-12] implicit methods call for a substantially greater deal of computational

Address for correspondence:
Eziokwu C. Emmanuel
E-mail: okereemm@yahoo.com

efforts than the explicit method whereas, on the other hands, for a given step number, implicit methods can be made more accurate than the explicit ones. Moreover, they have favorable stability properties as will be seen in Section 3.

MAIN RESULT ON LINEAR MULTI-STEPITERATIVE METHODS:

Analytical study of the linear multistep methods has revealed the following facts:-

a. That the domain of existence of solution of the linear multistep methods is the complete metric space.
b. That the solution of the linear multistep method converges in the complete metric space.
c. That the initial value problem $x' = f(t,x); x(t_0) = x_0$ solvable by the linear multistep in the complete metric space is a continuous function.
d. That the linear multistep method satisfies the conditions of the Banach contraction mapping principle.
e. That the linear multistep method is exactly the Picard's iterative method with a differential operator instead of the usual integral operator.

Theorem 2.1: Let X be a complete metric space and let R be a region in (t,x) plane containing(t_0,x_0) for $x_0, x \in X$. Suppose, given

$$\dot{x} = f(t,x); x(t_0) = x_0 \ldots \quad (2.0)$$

A differential equation where $f(t,x)$ is continuous. If the map f in (2.1) is Lipschitzian and with constant $K<1$, then the initial value problem (2.1) by the linear multistep method has a unique fixed point.

$$x^* = x_{n+k} = \sum_{j=0}^{k-1} \alpha_j x_{n+j} + h \sum_{j=0}^{k} \beta_j f_{n+j} \ldots$$

With a two-step predictor-corrector method.

$$\text{i.e.The predictor}: x_{j+1}^{(j)} = \sum_{j=0}^{n} \alpha_j x_j + h \sum_{j=0}^{n} \beta_j f_j$$

$$\text{The corrector}: x_{j+1}^{(j)} = \sum_{j=0}^{n} \alpha_j x_{j+1}^{(j)} + h \sum_{j=0}^{n} \beta_{j+1} f_{j+1}$$

For x_{n+k} satisfying. $h < \frac{1}{\ell h|^2{}_k|}, m = \ell h|^2{}_k|$

Proof

Let $x_1 = f(x_0)$

$$x_2 = f(x_1) = f(f(x_0)) = f^2(x_0)$$

$$x_3 = f(x_2) = f(f^2(x_0)) = f^3(x_0)$$

$$x_n = f(x_{n-1}) = f(f^{n-1}(x_0)) = f^n(x_0)$$

$$x_{n+k} = f(x_{n+k-1}) = f(f^{n+k-1}(x_0)) = f^{n+k}(x_0) \ldots \quad (2.1)$$

We have constructed a sequence $\{x_n\}_{n=0}$ in (X,ρ). We shall prove that this sequence is Cauchy.
First, we compute

$\rho(x_{n+k}, x_{n+k+1}) = \rho(f(x_{n+k}), f(x_{n+k+1}))$ Using (2.1)

$\leq K\rho(x_{n+k-2}, x_{n+k-1})$ Since is a contraction

$= K\rho(f_{n+k-2}, f_{n+k-1})$ Using (2.1)

$\leq K[K\rho(x_{n+k-2}, x_{n+k-1})]$ Since is a contraction

$= K^{2\,\rho}(x_{n+k-2}, x_{n+k-1})$

$K^{n+k}\rho(x_0, x_1)$

$$\text{i.e.}K\rho(x_{n+k}, x_{n+k+1}) \leq K^{n+k}\rho(x_0, x_1) \ldots \quad (2.2)$$

We can now show that $\{x_{n+k}\}_{n=0}$ is Cauchy.
Let $m+k > n+k$. Then

$$\rho(x_{n+k}, x_{m+k}) \leq \rho(x_{n+k}, x_{m+k}) + \rho(x_{n+k-1}, x_{m+k-2}) + \ldots + (x_{n+k-1}, x_{m+k})$$

$$\leq K^{n+k}\rho(x_0, x_1) + K^{n+k-1}\rho(x_0, x_1) + \ldots + K^{n+k-1}\rho(x_0, x_1) \quad \text{Using (2.2)}$$

$$= K^{n+k}\rho(x_0, x_1)(1 + k + k^2 + \ldots + k^{n+m-1} + k^{n+k} + \ldots)$$

Since the series on the right hand side is a geometric progression with common ratio <1, it sum to infinity is $\frac{1}{1-k}$. Hene, we have from above that

$$\rho(x_n, x_m) \leq k^{n-k}\rho(x_0, x_1)\left(\frac{1}{1-k}\right) \to 0$$

$$\text{as } n - k \to \infty \text{ since } k < 1$$

Hence, the sequence $\{x_{n+k}\}_{n=0}$ is a Cauchy sequence in X and since X is complete,

$\{x_{n+k}\}_{n=0}^{\infty}$ Converges to a point in X.]

$$\text{Let } x_{n+k} \to x^* \text{ as } h \to \infty \ldots \quad (2.3)$$

Since f is a contraction and hence is continuous it follows from (2.3) that

$f(x_{n+k}) \to f(x^*)$ as $n \to \infty$. But $f(x_{n+k}) = x_{n+k+1}$ from (2.2). So

$$x_{n+k+1} = f(x_{n+k}) = f(x^*)\ldots \quad (2.4)$$

However, limits are unique in a metric space, so from (2.3) and (2.4), we obtain that

$$f(x^*) = x^*\ldots \quad (2.5)$$

Hence, f has a unique fixed point in *X*. We shall now prove that this fixed point is unique. Suppose for contradiction, there exists $y^* \in X$ such that

$$y^* = x^* \text{ and } (y^*) = y^*\ldots \quad (2.6)$$

Then, from (2.5) and (2.6)

$$\rho\left(x^*, y^*\right) = \rho\left(f\left(x^*\right), f\left(y^*\right)\right) \le k\rho(x^*, y^*)$$

So that

$(k - 1)\ \rho(x^*, y^*) \ge 0$ and $\rho(x^*, y^*) = 0$.

We can divide by it to get $k-1 \ge 0$, that is, $k \ge 1$ which is a contradiction.

Hence $x^* = y^*$ and the fixed point is unique.

Therefore,

$$x_{n+k} = \sum_{j=0}^{k-1} \alpha_j x_{n+j} + h\sum_{j=0}^{k} \beta_j f_{n+j}; a \le t_n \le n$$

Is the linear multistep fixed point iterative formular for the initial value problem

$$\dot{x} = f\left(t, x\right); x\left(t_0\right) = x_0$$

Of the ordinary differential type.

Finally, to be sufficiently sure, we also show that

$$x^* = x_{n+k} = \sum_{j=0}^{k-1} \alpha_j x_{n+j} + h\sum_{j=0}^{k} \beta_j f_{n+j}$$

Satisfies the lipschitz condition.

$$\left|x^* - y^*\right| = \left|x_{n+k} - y_{n+k}\right|$$

$$= \left|\left(\sum_{j=0}^{k-1} \alpha_j x_{n+j} + h\sum_{j=0}^{k} \beta_j f_{n+j}\right) - \left(\sum_{j=0}^{k-1} \alpha_j y_{n+j} + h\sum_{j=0}^{k} \beta_j f^*_{n+j}\right)\right|$$

$$\le \sum_{j=0}^{k-1} \alpha_j \left|x_{n+j} - y_{n+j}\right| + h\sum_{j=0}^{k} \beta_j \left|f_{n+j} - f^*_{n+j}\right|$$

$$= k_1 \sum_{j=0}^{k-1} \left|x_{n+j} - y_{n+j}\right| + k_2 \sum_{j=0}^{k} \left|f_{n+j} - f^*_{n+j}\right|$$

$$= \left(k_1 + k_2\right)\left|\sum_{j=0}^{k-1} z_{n+j} + \sum_{j=0}^{k} f_{m+k}\right|$$

$$= K\sum_{j=0}^{k} \left|z_{n+j} + f_{m+j}\right|$$

Hence, $x^* = x_{n+k}$ is Lipschitizian and is a continuous map with the above fixed point.

Also in the same pattern, iterative methods for the respective linear multistep methods are as follows:

A. The Explicit Methods are:-

i. Euler:

$x_{n+1} = x_n + hf_n$

ii. The midpoints method:

$x_{n+2} = x_n + 2hf_{n+1}$

iii. Milne's method:

$$x_{n+1} = x_{n-3} + \frac{4h}{3}\left[2f_{n-2} + f_{n-1} + 2f_n\right]$$

iv. Adam's method:

$$x_{n+1} = x_n + \frac{h}{24}\left[55f_n - 59f_{n-1} + 35f_{n-2} - 9f_{n-3}\right]$$

v. The Generalized predictor method:

$$x_{j+1}^{(j)} = \sum_{j=0}^{n} \alpha_j x_j + h\sum_{j=1}^{n} \beta_j f_j$$

B. The Implicit Methods are:-

i. Trapezoidal method:

$$x_{n+1}^{(j+1)} = x_n + \frac{h^2}{6}\left[f_n + f_{n+1}^{(j)}\right]$$

ii. Simpson's method:

$$x_{n+2}^{(j+1)} = x_n + \frac{h}{3}\left[f_{n+1}^{(j)} + 4f_{n+1} + f_n\right]$$

iii. Simpson's method:

$$x_{n+2}^{(j+1)} = x_{n-2} + \frac{h}{3}\left[f_{n-1} + 4f_n + f_{n+1}^{(j)}\right]$$

iv. Adams Moulton's method

$$x_{n+2}^{(j+1)} = x_n + \frac{h}{3}\left[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right]$$

v. Milne's corrector method:

$$x_{n+1}^{(j+1)} = x_{n-2} + \frac{h}{3}\left[f_{n-1} + 4f_n + f_{n+1}^{(j)}\right]; n = 1$$

vi. The Generalized corrector method

$$x_{j+1}^{(j)} = \sum_{j=0}^{n} \alpha_j x_{j+1}^{(j)} + h\sum_{j=1}^{n} \beta_{j+1} f_{j+1}$$

C: The Generalized Iterative (Corrector – Predictor) Methods are

$$x_{j+1}^{(j)} = \sum_{j=0}^{n} \alpha_j x_j + h\sum_{j=1}^{n} \beta_j \ldots \quad (C1)$$

$$x_{j+1}^{(j)} = \sum_{j=0}^{n} \alpha_j x_{j+1}^{(j)} + h\sum_{j=1}^{n} \beta_{j+1} f_{j+1} \ldots \quad \text{(C2)}$$

Here, $x_{j+1}^{(j+1)} \in X$ is the corrector points to be determined for all j ≥ 0 while $x_{j+1}^{(j+1)} \in X$ is predetermined before $x_{j+1}^{(j+1)} \in X$. While the iterations are alternatively implement one after the other starting first with the predictor.

Note: The generalized compact form of C1 and C2 is as follows

$$x_{n+k-1} = \sum_{j=0}^{k-1} \alpha_j x_{n+j} + h\sum_{j=0}^{k} \beta_j f_{n+j}$$

Convergence Analysis

Given the linear multistep method

$$x_{n+k} = \sum_{j=0}^{k-1} \alpha_j x_{n+j} + h\sum_{j=0}^{k} \beta_j F_{n+j}; a \le t_n \le n \ldots \quad (2.7)$$

Theorem 2.2:[13-16]

Let $x_{n+j}=x(t_{n+j})$; j = 0,1,2,…,k–1 denote its numerical solution

$$T_{n+k} = \sum_{j=0}^{k-1} \alpha_j x\left(t_{n+j}\right) - h\sum_{j=0}^{k} \beta_j f(t_{n+j}, \left(x_{n+j}\right))$$

The local truncation error and

$$\tau_{n+k} = \frac{1}{h} T_{n+k}(x)$$

Then, the linear multistep method (3.1) is said to be consistent if

$\tau(h)=\max|T_{n+k}(x)| \to 0$ as $h \to 0$ and $\sum(h) = 0(h^m)$

For some m ≥ 1 or equivalently (1.3.2) is said to be consistent if

$$\sum_{j=0}^{k} \alpha_j; \sum_{j=0}^{k} j\alpha_j + \sum_{j=1} \beta_j = 1 \ldots \quad (2.8)$$

Proof:

If the numerical solution of a given linear multistep method is

$$x_{n+j} = x(t_{n+j}), j = 0,1,2,\ldots,k-1 \ldots \quad (2.9)$$

Moreover, the local truncation error is

$$T_{n+k} = \sum_{j=0}^{k-1} \alpha_j x\left(t_{n+j}\right) - h\sum_{j=0}^{k} \beta_j f\left(t_{n+j}, x_{n+j}\right) \ldots \quad (2.10)$$

$$\text{With } \tau_{n+k}(x) = \frac{1}{h} T_{n+k}(x), \ldots \quad (2.11)$$

We want to prove that the linear multistep method

$$x_{n+k} = \sum_{j=0}^{k-1} \alpha_j x_{n+j} + h\sum_{j=0}^{k} \beta_j f_{n+j}, a \le t_{n+j} \le n \quad (2.12)$$

Is consistent if

$$\tau(h) = \max\left|T_{n+k}(x)\right| \to 0 \text{ as } h \to 0 \ldots \quad (2.13)$$

And

$\tau(h)= 0(h^m)$ For some m ≥ 1… (2.14)

If $\bar{x}_{n+j}$ denotes the numerical solution with the above exact values (1.2), then (2.14) yields

$$x_{n+k} + \sum_{j=0}^{k-1} \alpha_j x_{n+j} = h\beta_k f\left(t_{n+k}, \bar{x}_{n+k}\right) + h\sum_{k=0}^{k-1} \beta f\left(t_{n+j}, x_{n+j}\right) \ldots \quad (2.15)$$

Applying localizing assumption on (2.15) means that no previous truncation error has been made and that

$$x_{n+j} = x\left(t_{n+j}\right), j = 0,1,\ldots,k-1$$

So that we have

$$\bar{x}_{n+k} + \sum_{j=0}^{k} \alpha_j \bar{x}(t_{n+j}) = h\beta_k f\left(t_{n+k}, \bar{x}_{n+k}\right)$$
$$+ h\sum_{k=0}^{k-1} \beta_j f\left(t_{n+j}, \bar{x}_{n+j}\right) \ldots \quad (2.16)$$

Using the local truncation error earlier defined in (2.2), we now have

$$x(t_{n+k}) + \sum_{j=0}^{k-1} \alpha_j x(t_{n+j})) = T_{n+k} + h\beta_k f\left(t_{n+k})\right), x(t_{n+k})$$
$$+ h\sum_{k=0}^{k-1} \beta_j f\left(t_{n+j}, x(t_{n+j})\right) \ldots \quad (2.17)$$

Subtracting (2.16) from (2.17) we have

$$x(t_{n+k}) - \bar{x}_{n+k} = T_{n+k} + h\beta_k [f\left(t_{n+k}, x(t_{n+k})\right)$$
$$- f\left(t_{n+k}, \bar{x}_{n+k}\right) \ldots \quad (2.18)$$

If we apply mean value theorem on (3.1.10),
We have

$$f\left(t_{n+k}, x(t_{n+k})\right) - f\left(t_{n+k}, \bar{x}_{n+k}\right) =$$
$$\left(x\left(t_{n+k}\right) - \bar{x}_{n+k}\right) \frac{\partial f}{\partial x} \Big| (t_{n+k,} \tau_{n+k})$$

Where η_{n+k} lies between $\bar{x}_{n+k}$ and $x(t_{n+k})$

Therefore,

$$\left[1 - h\beta_k \frac{\partial f}{\partial x}\left|\left(t_{n+k}, \eta_{n+k}\right)\right]\left(x\left(t_{n+k}\right), \bar{x}_{n+k}\right) = T_{n+k} \ldots\right. \quad (2.19)$$

Let e_{n+k} represent the error at (n+k) point, so that if the method is explicit β_k=0 and then T_{n+k}=e_{n+k} but if the method is implicit $\beta_k \neq 0$ and

$h\beta_k \left(\frac{\partial f}{\partial x}\right)\left|\left(t_{n+k}, n_{n+k}\right)\right.$ is small then $T_{n+k} \approx e_{n+k}$

Again let

$$\tau_{(n+k)}(x) = \frac{1}{h} T_{n+k}(x) \ldots \quad (2.20)$$

For us to show that the approximate solution $\{x_n | t_0 \leq t_n \leq b\}$ ┤ of (3.1.4) converges to the theoretical solution x(t) of the initial value problem

$$\dot{x} = f(t, x); x(t_0) = x_0$$

We[13,14] need to necessarily satisfy the consistency condition

$$\tau(h) = \max_{t_0 \leq t_n \leq b} \left|T_{n+k}(x)\right| \to 0 \text{ as } h \to 0 \ldots \quad (2.21)$$

Plus the condition that

$\tau(h) = 0(h^m)$, for some $m \geq 1\ldots$ (2.22)

By this, we[15] show the only necessary and sufficient condition for the linear multistep (2.14) to be consistent is that

$$\sum_{j=0}^{k} \alpha_j = 1 \text{ and} - \sum_{j=0}^{k} j\alpha_j + \sum_{j=1}^{k} \beta_j = 1 \ldots \quad (2.23)$$

And for (2.23) above to be valid for all functions x(t) is for x(t) that are m + 1 times continuously differentiable to necessarily satisfy

$$\sum_{j=0}^{k} (-j)\alpha_j + \sum_{j=1}^{k} (-j)^{i-1}\beta_j = 1, i = 2, \ldots m \ldots \quad (2.24)$$

Hence, we know that

$$T_{n+k}(\alpha x + \beta w) = \alpha T_{n+k}(x) + \beta T_{n+k}(w) \ldots \quad (2.25)$$

For all constants α, β and all differentiable functions x,w. we now examine the consequence of (2.19) and (2.20) by expanding x(t) about t_n using Taylor's theorem and we have

$$x(t) = \sum_{j=o}^{k} \frac{1}{j}(t - t_0) x^1(t_n) + R_{m+1}(t) \quad (2.26)$$

Assuming x(t) is m+1 times continuously differentiable. Substituting into the truncation error

$$T_{n+k}(x) = x(t_{n+k}) - \sum_{j=0}^{k} \alpha_j x(t_{n+j}) + h \sum_{j=1}^{k} \beta_j F(t_{n+j}) \ldots \quad (2.27)$$

And also using (2.23)

$$T_{n+k}(x) = \sum_{j=0}^{m-1} \frac{1}{j} x^{(j)}(t_n) T_{n+k}(((t - t_n)^j) + T_{n+k}(R_{m+1}) \ldots \quad (2.28)$$

It becomes necessary[16,17]; Keller[18] to calculate T_{n+k} $(t-t_n)^j$ For j=0

$$T_{n+k}(1) = c_0 \equiv 1 - \sum_{j=0}^{k} \alpha_j \ldots \quad (2.29)$$

For $j \geq 1$ we have

$$T_{n+k}(t - t_n)^j = T_{n+k}(t - t_n)^j$$

$$= \left(\sum_{j=0}^{k} \alpha_j (t_{n+k} - t_n)^j + h \sum_{j=0}^{k} \beta_j^i (t_{n+j} - t_n)^{i=1}\right) = c_1 h^1 \ldots \quad (2.30)$$

$$C_j = 1 - \left(\sum_{j=0}^{k} (-j)^i \alpha_j + i \sum_{j=1}^{k} (-j)^{i=1} \beta_j\right), i \geq 1$$

This gives

$$T_{n+k}(x) = \sum_{j=1}^{m} \frac{c}{j} h^j x^{(j)}(t_n) + T_{n+k}(R_{m+1}) \ldots \quad (2.31)$$

Moreover, if we write the remainder R_{m+1} (t) as

$$R_{m+1}(t) = \frac{1}{(m+1)!}(t - t_n)^{m+1} x^{m+1}(t_n) + \ldots$$

Then,

$$T_{n+k} R_{m+1}(t) = \frac{C_{m+1}}{(m+1)!} h^{m+1} x^{m+1}(t_n) + 0(h^{m+2}) \ldots \quad (2.32)$$

To obtain the consistency condition (2.20), we need $\tau(h)$– 0(h) and this requires T_{n+k} $(x)=0(h^2)$. Using (2.19) with m=1, we must have C_0,C_1=0 which gives the set of equations (2.22) which are referred to as consistency conditions in some texts. Finally, to obtain (3.1.14) for some $m \geq 1$, we must

have $T_{n+k}(x)=0(h^{m+1})$. From (2.30) and (2.31), this will be true if and only if $C_i=0, i=0,1,2\ldots m$.
This proves the conditions (2.21) and completes the proof.

Theorem 2.[17-20] Assume the consistency condition of (2.10), then the linear multistep method (1.2) is stable if and only if the following root conditions (2.11)-(2.12) are satisfied

$$\text{The root } |r_j|< 1 = j = 0,1,\ldots,k\ldots \quad (2.33)$$

$$|r_j|=1 \Rightarrow \rho^1(r_j) \neq 0\ldots \quad (2.34)$$

Where

$$\rho(r) = r^{k-1} - \sum_{j=0}^{k} \alpha_j r^j$$

Proof:
Given the linear multistep

$$x_{n+k} + \sum_{j=0}^{k-1} \alpha_j x_{n+j} + h\sum_{k=0}^{k-1} \beta_j f_{n+j}; a \le t_{n+j} \le n\ldots \quad (2.35)$$

With the associated characteristic polynomial

$$P(r) = r^{k+1} - \sum_{j=0}^{k} \alpha_j r^j \ldots \quad (2.36)$$

such that P(1)=0 by the consistency condition. Let $r_0,\ldots,r_n$ denote the respective roots of P(r), repeated according to their multiplying and let $r_0=1$.
The linear multistep method 2.8. Satisfies the root condition if

$$|r_j| \le 1, j=0,1,\ldots,k\ldots \quad (2.37)$$

$$|r_j|=1 \Rightarrow P^1(r_j) \neq 0\ldots \quad (2.38)$$

Let (2.8) be stable we now prove that the root condition (2.37) and (2.38) are satisfied. By contradiction let
$|r_j(0)|>1$ for some j. This is to say we consider the initial value problem x^1≡0;x(0)=0 with solution x(t)=0. So that (2.8) becomes

$$x_{n+k} = \sum_{j=0}^{k-1} \alpha_j x_{n+j}; n \ge k\ldots \quad (2.39)$$

If we take $x_0=x_1=...=x_k=0$, then the numerical solution clearly becomes $x_n=0$ for all $n \ge 0$.
For perturbed initial values, let

$$z_0=\in, z_1=\in r_1(0),\ldots,z_n=\in r_1(0)^p\ldots \quad (2.40)$$

And for these initial values

$$\max_{0\le n-k} |x_n - z_n| \le \in |r_1(0)|^p$$

Which is a uniform bound for all small values of h, since the right side is independent of h, as $\in\to 0$, the bound also tend to zero.
The solution (2.12)[22-24] with the initial condition (2.13) is simply $z_n=\in r_j(0)^n; n\ge 0$. For the derivation from $\{x_n\}$

$$\max_{0\le n-k} |x_n - z_n| = N(h) \to \infty$$

Moreover, the bound that the method is unstable when $|r_j(0)| > 0$. Hence, if the method is stable, the root condition $|r_j(0)| \le 1$ must be satisfied.
Conversely, assume the root condition is satisfied, we now prove for stability restricted to the exponential equation.

$$x^1=\lambda x;\ x(0)=1. \quad (2.41)$$

This[25-27] involves solution of non-homogenous linear difference equations which we simplify by assuming the roots $r_j(0); j=0,1,\ldots,k$ to be distinct. The same will be true of $r_j(h\lambda)$ provided the values of h is kept sufficiently small, say $0\le h \le h_0$. Assume $\{x_n\}$ and $\{z_n\}$ to be two solutions of

$$(1-h\lambda\beta_{k+1})x_{n+k+1} - \sum_{j=0}^{k-1}(\alpha_j + h\lambda\beta_j)x_{n+j} = 0; n \ge 1\ldots \quad (2.42)$$

On (2.10) on $[x_0,b]$ and assume that

$$\max_{0\le n-k} |x_n - z_n| \le \in, 0 \le h \le h_0$$

Introduce the error $e_n = x_n - z_n$ and subtracting using (2.3.8) for each solution

$$(1-h\lambda\beta_k)e_{n+k} - (\alpha_j + h\lambda\beta_j)e_j = 0; x_{k+1} \le x_{n+k} \le b\ldots \quad (2.43)$$

The general equation becomes

$$e_n = \sum_{j=1}^{k} \gamma_j |r_j(h\lambda)|^n; n \ge 0 \quad (2.44)$$

The coefficient $\gamma_0,\ldots,\gamma_k$ must be chosen so that the solution (2.17) will then agree with the given initial perturbations $e_0,\ldots,e_k$ and will satisfy the difference equation (2.16). Using the bound $z_0=\in, z_1=\in r_1(0),\ldots,z_n=\in r_j(0)^p$ and the theory of linear system of equations, we have

$$\max_{0\le n-k} |\gamma_n| \le c\varepsilon; 0 \le h \le h_0\ldots \quad (2.45)$$

for some constants $c_j>0$.
To bound the solution e_n on $[x_0,b]$, we must bound each term $[r_j(h\lambda)]^n$ to do so, consider the expansion

$(U) = r_j(0) + U r_j(\xi)\ldots$ (2.46)

For some ξ between 0 and U. To compute $r_j^1(u)$, differentiate the identity

$$p\left(r_j(u)\right) - u\sigma\left(r_j(u)\right) = 0$$

$$p^1\left(r_j(cu)\right) - r_j^1(u) - \left[\sigma_j(u) + u\sigma^1\left(r_j(u)\right)\left(r^1(u)\right)\right] = 0$$

$$r_j^1(u)[p^1\left(r_j(u)\right) - u\sigma^1\left(r_j(u)\right) = \sigma(r_j(u))$$

$$r_j^1(u) = \frac{\sigma\left(r(u)\right)}{p\left(r_j(u)\right) - ur^1(r(u))}\ldots \quad (2.47)$$

By assumption that $r_j(0)$ is a simple root of $p(r)=0$; $0 \le j \le k$, it follows that $p^1(r_j(0))=0$ and by continuity, $p^1(r_j(u)) \neq 0$ for all sufficiently small values of u, the denominator in (2.20) is non-zero and we can bound $r_j(u)|r_j(u)|=c_2$ for all $|u| \le u_0$ For some $U_0 \ge 0$.

Using this with (3.3.12) and the root condition (3.2.4), we [Stetter;[28] Stetter[29]] have

$$\left|[r_j(h\lambda)]^n\right| \le \left|r_j(0)\right| + c_2\left|(h\lambda)\right| \le 1 + c_2\left|(h\lambda)\right|$$

$$\left|[r_j(h\lambda)]^n\right| \le [1 + c_2\left|(h\lambda)\right|]^n \le e^{C_{2h|h\lambda|}} \le e^{c_2}(bx_n\ )\ |\ \lambda\ |$$

for all $0 \le h \le h_0$.

Combine this with (2.18) and (2.19) to get $\text{Max}|e_n| \le c_2 \le |\varepsilon|\ e^{c2}\ (bx_n)\ |\lambda|$ for an approximate constant c_0. This concludes the proof.

Theorem2.3:[21-24] The linear multistep method (1.2) is said to be convergent if and only if it is consistent and stable.

Proof:

By this, we want proof that if the consistency condition is assumed, the linear multistep method

$$x_{n+k} = \sum_{j=0}^{k-1}\alpha_j x_{n+j} + h\sum_{j=0}^{k}\beta_j f_{n+j}; a \le t_{n+j} \le n \quad (2.48)$$

Is convergent if and only if the root conditions (2.10) and (2.11) are satisfied.

We assume first the root conditions are satisfied and then show the linear multi step (2.8)

Is convergent. To start, we use the problem x=0, x(0) =0 with the solution x (t) = 0. Then, the multi-step method (2.8) becomes

$$x_{n+k} = \sum_{j=0}^{k-1}\alpha_j x_{n+j}, n \ge k\ldots \quad (2.49)$$

With $x_0,\ldots,x_k$

Satisfying $n(h) = \max|x_n| \to 0$ as $h \to 0\ldots$ (2.50)

Suppose[25-28] that the not condition is violated, we will show that (2.10) is not convergent to x(t)=0. Assume that some $|r_j(0)|>1$ then a satisfactory solution of (2.11) is

$$x_n = h[r_j(0)]^n; t_0 \le t_n \le b\ldots$$

Condition (2.10) is satisfied since $n(h)=|h(r_j(0))| \to 0$ as $h \to 0$.

However, the solution (2.11) does not converge. First

$$\text{Max}\left|x(t_n) - x_n\right| = h\left|h[r_j(0)\right|^{N(h)}\ 0 \le t_n \le b$$

Consider those values of $h = \frac{b}{N(h)}$. Then, L' Hospital's rule can be used to show that

$$\text{Lim}\,\frac{b}{N}\left|r(0)\right|^N = \infty$$

Showing that (2.11) does not converge.

Conversely assume the root condition is satisfied as with theorem 2.2; it is rather difficult to give a general proof of converge for an arbitrary differential equation. The present proof is restricted to the exponential equation (2.14) and again we assume that the roots $r_j=0$ are distinct.

To simplify the proof, we will show that the term $\gamma_0\ [r_0(\lambda\lambda)]^n$ in the solution

$$x_n = \sum_{j=0}^{k}\gamma_j\left|r_j(h\lambda)\right|^n$$

Will converge to the solution $e^{\lambda t}$ on [0,b]. The remaining terms

$\gamma_j|r_j(h\lambda)|^n, j=1,2,\ldots,k$ are parasitic solution to converge to zero as $h \to 0$. Expand r_0 $(h\lambda)$ using Taylor's theorem,

$$r_0(h\lambda) = r_0(0) + h\lambda r_0^2(0) + 0(h^2)$$

From (2.19) $r_0^2(0) = \frac{\sigma(1)}{\rho^1(1)}$ and using this consistency condition (2.11), this leads to $r_0^2(0)=1$. then

$$r_0(h\lambda) = 1 + h\lambda + 0\left(h^2\right) = e^{h\lambda} + 0(h^2)$$

$$[r_0(h\lambda)]^n = e^{hn\lambda}[1 + 0\left(h^2\right)]^n = e^{\lambda tn}[1 + 0\left(h^2\right)]$$

Thus,

$$\max_{0 \le t_n \le b}\left|[r_0(h\lambda)] = e^{\lambda t_n}\ \right| \to 0\, as\, h \to 0\ldots \quad (2.51)$$

We[30,31] must now show that the coefficient $\gamma_0 \to 1$ as h→1. The coefficient $\gamma_0,\ldots,\gamma_k$ satisfy the linear system

$$\gamma_0+\gamma_1+\ldots+\gamma_k=x_0$$

$$\gamma_0\left[r_0(h\lambda)\right]+\ldots+\gamma_k[r_k(h\lambda)]=x_1$$

$$\gamma_0\left[r_0(h\lambda)\right]^k+\ldots+\gamma_k[r_k(h\lambda)]^k=x_2\ldots \quad (2.52)$$

The initial values $x_0,\ldots,x_k$ are assumed to satisfy

$$r_j(h)\max_{0\le n-k}\left|e^{\lambda t_n}-x_n\right|\to 0 \text{ as } h\to 0$$

However, this implies

$$\text{Lim } x_n=1, 0\le n\le p\ldots \quad (2.53)$$

The coefficient γ_0 can be obtained using Cramer's rule to solve (3.3.5) then

$$\gamma_0=\frac{\begin{vmatrix} x_0 & 1 & \cdots & 1 \\ x_1 & r_1 & \cdots & r_k \\ \vdots & & & \\ x_k & r_k & \cdots & r_k^k \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_0 & r_1 & \cdots & r_k \\ \vdots & & & \\ r_k^k & r_1^k & \cdots & r_k^k \end{vmatrix}}$$

The denominator converges to the vandermonde determinant for $r_0(0)=1, r_1(0),\ldots,r_k(0)$; and this is non-zero since the roots are distinct. Using (2.13), the numerator converges to the same quantity as $h\to 0$. Therefore, $\gamma\to 1$ as $h\to 0$, using this, along with (2.10), the solution $\{x_n\}$ converges to $x(t)=e^{\lambda t}$ on [0,b]. This completes the proof.

ILLUSTRATIVE EXPERIMENT

Example 3.1 (problem statement)

Write a general-purpose function named HAMMING that solves a system offirst-order ordinary differential equations.

$$\frac{dy_1}{dx}=f_1(x,y_1,y_2,\ldots,y_n),$$

$$\frac{dy_2}{dx}=f_2(x,y_1,y_2,\ldots,y_n) \quad (3.1)$$

$$\frac{dy_n}{dx}=f_n(x,y_1,y_2,\ldots,y_n),$$

Using Hamming's predictor-corrector method embedded in C_1 and C_2 of page 6, Section 2 and also in,[3] write a main program that solves the second-order ordinary differential equation.

$$\frac{d^2y}{dx^2}=-y, \quad (3.2)$$

Subject to the initial conditions

$$y(0)=0 \quad \frac{dy}{dx}(0)=1 \quad (3.3)$$

The program should call on the fourth-order Runge-kutta function RUNGE, see[3] to find the essential starting values for Hamming's algorithm as

$$y_{1,0}=V_o \quad (3.4)$$

$$y_{2,0}=0$$

And thereafter, should call on HAMMING to calculate estimates of y and dy/dx on the interval $[0,x_{max}]$. Evaluate the numerical solutions for several different step sizes, h, and compare the numerical results with the true solutions.

$$y(x)=\sin x \quad (3.5)$$

$$\frac{dy}{dx}=\cos x.$$

Method of Solution

Let $y_{j,i}$ be the final modification of the estimated solution for the j th dependent variable, y_j, x_i, resulting from Hamming's method and left $f_{j,i}$ be the calculated estimate of f_j at x_i, that is,

$$y_{j,i}=y_j(x_i),$$

$$f_{j,i}=f_j(x_i,y_{1,i},y_{2,i},\ldots,y_{n,i}). \quad (3.6)$$

Assume that $y_{j,i}, y_{j,i-1}, y_{j,i-2}, y_{j,i-3}$, $f_{j,i}, f_{j,i-1}$, $f_{j,1-2}$, have already been found and are available for $j=1,2,\ldots,n$. Let an estimate of the local truncation error for the corrector equation (3.4) for the j th dependent variable be denoted $e_{j,i}$. Then, assuming that the $e_{j,i}$, $j=1,2,\ldots,n$, are available, the procedure outlined for Hamming's method in the above may be modified to handle a system of n simultaneous first-order ordinary differential equations by simply appending a leading subscript, j, to all if all y and f terms in (3.1) to (3.6). In terms of the new nomenclature, Steps 2 through 6 of the outline in the above are:

2. The predicted solutions are computed using the Milne predictor below;

$$y_{j,i+1,0} = y_{j,i-3} + \frac{4}{3} hf(2f_{j,i} - f_{j,i-1} + 2f_{j,i-2}), j = 1, 2, \ldots, n \tag{3.7}$$

3. The predicted solutions, $y_{j,i+1,0}$, are modified (assuming that the local truncation error estimates, $e_{j,i+1}$, *j*=1,2,…,n, will not be significantly different from estimates $e_{j,i}$, j=1,2,…,n) as in (3.5):

$$Y_{j,i+1,0} = y_{j,i+1,0} + \frac{112}{9} e_{j,i}, \quad j = 1, 2, \ldots, n. \tag{3.8}$$

4. The *j* th corrector equation corresponding to (3.4) is applied for each dependent variable:

$$y_{j,i+1,1} = +\frac{1}{8}\left[9y_{j,i} - y_{j,i-2} + 3h\left(f_{j,i+1,0} + 2f_{j,i-1}\right)\right], \quad j = 1, 2, ..., n \tag{3.9}$$

Where

$$f_{j,i+1,0} = f(x_{i+1}, y_{1,i+1,0}, \ldots, y_{2,i+1,0}, y_{n,i+1,0}) \tag{3.10}$$

The corrector in (3.5) is being applied just once for each variable, the customary practice. The corrector equations could, however, be applied more than once, as in (3.6). Note that the subscript *j,* in (3.6) is an iteration counter, and is not the index *j*, on the dependent variables used throughout this example.

5. Estimate the local truncation error for each of the corrector equations on the current interval as in (3.6)

$$e_{j,i+1} = \frac{9}{121}\left(y_{j,i+1,1} - y_{j,i+1,0}\right), \quad j = 1, 2, \ldots, n. \tag{3.11}$$

6. Make the final modifications of the solutions of the corrector equations as in (3.8)

$$y_{j,i+1} = y_{j,i+1,1} - e_{j,i+1}, \quad j = 1, 2, \ldots, n \tag{3.12}$$

After evaluation the $y_{j,i+1,}$ the *n* values $f_{j,i+1}$ may be computed from (3.13) as

$$f_{j,i+1} = f_j\left(x_{i+1,} y_{1,i+1}, y_{2,i+1}, \ldots, y_{n,i+1}\right), \quad j = 1, 2, \ldots, n. \tag{3.13}$$

If desired, the entire process may be repeated for the next interval by starting again at Step 2. Therefore, the function HAMING can be written to solving arbitrary system of first-order equations, provided that the 5n+2 essential numbers, xi,h and $y_{j,i-3}$,

$$\left.\begin{matrix} f_{j,i},\ e_{j,i}, \\ f_{j,i-1}, \\ f_{j,i-2}, \end{matrix}\right\} J = 1, 2, \ldots, n, \tag{3.14}$$

Are available for the function to use on entry to the predictor section, and the essential numbers and

$$\left.\begin{matrix} y_{j,i},\ f_{j,i}, \\ y_{j,i-2},\ f_{j,i-1}, \\ y_{j,i+1,0,}\ f_{j,i-2}, \end{matrix}\right\} \quad j = 1, 2, \ldots, n, \tag{3.15}$$

Then, write the calling program and function HAMMING, so that the matrices Y and F, and the vector, e′ have the following contents at the indicated points in the algorithm.

Before entry into the corrector section of HAMMING:

Y is unchanged from (3.16)

$$F = \begin{bmatrix} f_{1,i+0} & f_{2,i+1,0} \cdots f_{j,i+1,0} \cdots f_{n,i+1,0} \\ f_{1,i} & f_{2,i} \ \cdots \ f_{j,i} \ \cdots \ f_{n,i} \\ f_{1,i-1} & f_{2,i-1} \ \cdots \ f_{j,i-1} \ \cdots \ f_{n,i-1} \end{bmatrix} \tag{3.16}$$

e′ is unchanged from (3.16)

After return from the corrector section of HAMMING:

$$Y = \begin{bmatrix} \\ y_{1,i+1} & y_{2,i+1} \cdots y_{j,i+1} \cdots y_{n,i+1} \\ y_{1,i} & y_{2,i} \ \cdots \ y_{j,i} \ \cdots \ y_{n,i} \\ y_{1,i-1} & y_{2,1-1} \cdots y_{j,i-1} \cdots y_{n,i-1} \\ y_{1,i-2} & y_{2,1-2} \cdots y_{j,i-2} \cdots y_{n,i-2} \end{bmatrix} \tag{3.17}$$

F is unchanged from (3.20)

$$e' = \left[e_{1,i} e_{2,i} \cdots e_{j,i} \cdots e_{n,i}\right] \tag{3.18}$$

If the calling program uses (3.14) to replace the elements in the first row of F after the return from the corrector section of HAMING, so that

$$F = \begin{bmatrix} f_{1,i+1} & f_{2,i+1} \cdots f_{j,i+1} \cdots f_{n,i+1} \\ f_{1,i} & f_{2,i} \ \cdots \ f_{j,i} \ \cdots \ f_{n,i} \\ f_{1,i-1} & f_{2,i-1} \cdots f_{j,i-1} \cdots f_{n,i-1} \end{bmatrix} \tag{3.19}$$

Then, the matrices Y and F and the vector e′ are ready for a call on the predictor section of HAMING for the next integration step. The independent variable is incremented in the predictor section, that is,

$$x_{i+1} = x_i + h \tag{3.20}$$

So that the calling program will automatically have the proper value to calculate the $f_{j,i+1,0}$ and $f_{j,i+1}$ from (3.10) and (3.14), respectively.
Assuming that the integration process begins at $x=x_0$ and that the only known conditions on (3.2) are

$$y_{1,0}=c_1$$
$$y_{2,0}=c_2 \quad (3.21)$$
$$y_{n,0}=c_n$$

There is not enough information to calculate the elements of (3.17), (3.18), and (3.19). Therefore, HAMMING cannot be called directly when $x_1=x_0$. The usual procedure is to use a one-step method to integrate across the first steps to evaluate

$$\left.\begin{matrix} y_{j,1} \\ y_{j,2} \\ y_{j,3} \end{matrix}\right\} \quad j=1,2,\ldots,n \quad (3.22)$$

With (3.15) and (3.16), the matrix Y of (3.17) is known for i=3. The matrix F of (3.19) may be evaluated from (3.15) and (3.16) for i=3. The vector of local truncation error estimates (3.14) for i=3 is normally unknown, and should be set to zero unless better values are available from other sources. Hamming may then be called for the first time.
In the calling program that follows, the function RUNGE, already developed in,[3] is used to generate the solutions of (3.22). Since RUNGE implements the fourth-order Runge-kutta method, the solutions of (3.27) should be comparable in accuracy with the solutions generated by the Hamming's predictor-corrector algorithm, also a fourth-order method.
The main program reads data values for $n, x_0, h, x_{max}, int, y_{1,0}, y_{2,0}, \ldots, y_{n,0}$. Here, int is the number of intergration steps between the printings of solution values. This program is a reasonably general one. However, the defining statements for computation of the derivative estimates (3.11) and (3.14) would be different for each system of differential equations.
For test purposes, the differential equation solved is (3.2), subject to the initial conditions of (3.3). First, the second-order equation must be written as a system or two first-order equations. Let
$y_1=y$

$$y_2=\frac{dy}{dx} \quad (3.23)$$

Then,

$$\frac{dy_1}{dx}=f_1(x,y_1,y_2)=\frac{dy}{dx}=y_2 \quad (3.24)$$

$$\frac{dy_2}{dx}=f_2(x,y_1,y_2)=\frac{d^2y}{dx^2}=-y=-y_1$$

So that

$$f_{1,i}=y_{2,i}$$
$$f_2=-y_{2,I} \quad (3.25)$$

The initial conditions of (3.3)

$$y_{1,0}=0$$
$$y_{2,0}=1 \quad (3.26)$$

With a known program listing, given, we use the listed data below
Data

X=	0.0000	H=	1.000000	XMAX=	5.0000
INT=	1	N=	2		
YR (1)	YR (2)=		0.00000	1.00000	
X=	0.0000	H=	0.500000	XMAX=	5.0000
INT= 1	N=	2			
YR (1)	YR (2)		0.00000	1.00000	
X=	0.00000	H=	0.250000	XMAX=	5.00000
INT=	2	N=	2		
YR (1)	YR (2)=	0.00000	1.00000		
X=	0.0000	H=	0.125000	XMAX=	5.0000
INT=	4	N=	2		
YR (1)	YR (2)		0.00000	1.00000	
X=	0.0000	H=	0.062500	XMAX=	5.0000
INT=	8	N=	2		
YR (1)	YR (2)=		0.00000	1.00000	
X=	0.0000	H=	0.031250	XMAX=	5.0000
INT=	16	N=	2		
YR (1)	YR (2)		0.00000	1.00000	
X=	0.00000	H=	0.015625	XMAX=	5.00000
INT=	32	N=	2		
YR (1)	YR (2)=		0.00000	1.00000	

Moreover, we have the following results

Computer Output

Results for the first data set
H = 0.100D 01
XMAX = 5.0000
INT = 1
N = 2
X Y (1) Y (2)

X	Y (1)		Y (2)	
0.0	0.0		0.1000000	01
1.0000	0.8333333D	00	0.5416667D	00
2.0000	0.9027778D	00	–0.4010417D	00
3.0000	0.1548032D	00	–0.9695457D	00
4.0000	–0.6638999D	00	–0.6880225D	00
5.0000	–0.1045312D	00	0.1429169D	00

Results for the second data set
H = 0.500D 00
XMAX = 5.0000
INT = 1
N = 2

X	Y (1)		Y (2)	
0.0	0.0		0.1000000D	01
0.5000	0.4791667D	00	0.8776042D	00
1.0000	0.8410373D	00	0.5405884D	00
1.5000	0.9971298D	00	0.7142556D	-01
2.0000	0.9104181D	00	–0.4146978D	00
2.5000	0.6001440D	00	–0.8012885D	00
3.0000	0.1422377D	00	–0.9912065D	00
3.5000	–0.3510641D	00	–0.9387887D	00
4.0000	–0.7588721D	00	–0.6554891D	00
4.5000	–0.9810034D	00	–0.2112528D	00
5.0000	–0.9627114D	00	0.2857746D	00

Results for the third data set
H = 0.250D 00
XMAX = 5.0000
INT = 2
N = 2

X	Y (1)		Y (2)	
0.0	0.0		0.1000000D	01
0.5000	0.4791667D	00	0.8776042D	00
1.0000	0.8410373D	00	0.5405884D	00
1.5000	0.9971298D	00	0.7142556D	-01
2.0000	0.9104181D	00	–0.4146978D	00
2.5000	0.6001440D	00	–0.8012885D	00
3.0000	0.1422377D	00	–0.9912065D	00
3.5000	–0.3510641D	00	–0.9387887D	00
4.0000	–0.7588721D	00	–0.6554891D	00
4.5000	–0.9810034D	00	–0.2112528D	00
5.0000	–0.9627114D	00	0.2857746D	00

Results for the fourth data set
H = 0.125D 00
XMAX = 5.0000
INT = 4
N = 2

X	Y (1)		Y (2)	
0.0	0.0	00	0.1000000D	-01
0.5000	0.4794249D	00	0.8775832D	00
1.0000	0.8414708D	00	0.5403028D	00
1.5000	0.9974948D	00	0.5403028D	-01
2.0000	0.9092971D	00	–0.4161466D	00
2.5000	0.5984717D	00	–0.8011433D	00
3.0000	0.1411195D	00	–0.9899918D	00
3.5000	–0.3507835D	00	–0.9364555D	00
4.0000	–0.7568022D	00	–0.6536421D	00
4.5000	–0.9775289D	00	–0.2107944D	00
5.0000	–0.9589223D	00	0.2836631D	00

Results for the fifth data set
H = 0.625D-01
XMAX = 5.0000
INT = 8
N = 2

X	Y (1)		Y (2)	
0.0	0.0	00	0.1000000D	-01
0.5000	0.4794255D	00	0.8775826D	00
1.0000	0.8414710D	00	0.5403023D	00
1.5000	0.9974950D	00	0.7073721D	-01
2.0000	0.9092974D	00	–0.4161468D	00
2.5000	0.5984721D	00	–0.8011436D	00
3.0000	0.1411200D	00	–0.9899925D	00
3.5000	–0.3507832D	00	–0.9364566D	00
4.0000	–0.7568025D	00	–0.6536436D	00
4.5000	–0.9775301D	00	–0.2107958D	00
5.0000	–0.9589242D	00	0.2836622D	00

Results for the sixth data set
H = 0.312D-01
XMAX = 5.0000
INT = 16
N = 2

X	Y (1)		Y (2)	
0.0	0.0	00	0.1000000D	–01
0.5000	0.4794255D	00	0.8775826D	00
1.0000	0.8414710D	00	0.5403023D	00
1.5000	0.9974950D	00	0.7073720D	–01
2.0000	0.9092974D	00	–0.4161468D	00
2.5000	0.5984721D	00	–0.8011436D	00
3.0000	0.1411200D	00	–0.9899925D	00
3.5000	–0.3507832D	00	–0.9364567D	00
4.0000	–0.7568025D	00	–0.6536436D	00
4.5000	–0.9775301D	00	–0.2107958D	00
5.0000	–0.9589243D	00	0.2836622D	00

Results for the seventh data set
H= 0.156D-01
XMAX= 5.0000
INT= 32
N= 2

X	Y (1)		Y (2)	
0.0	0.0	00	0.1000000D	–01
0.5000	0.4794255D	00	0.8775826D	00

1.0000	0.8414710D	00	0.5403023D	00
1.5000	0.9974950D	00	0.7073720D	–01
2.0000	0.9092974D	00	–0.4161468D	00
2.5000	0.5984721D	00	–0.8011436D	00
3.0000	0.1411200D	00	–0.9899925D	00
3.5000	–0.3507832D	00	–0.9364567D	00
4.0000	–0.7568025D	00	–0.6536436D	00
4.5000	–0.9775301D	00	–0.2107958D	00
5.0000	–0.9589243D	00	0.2836622D	00

DISCUSSION OF RESULTS

Double-precision arithmetic has been used for all calculations.

Differential equation (3.2) with initial conditions given by (3.2) has been solved on the interval [0,5] with step-size h=1.0, 0, 0.25, 0.125, 0.0625, 0.03125, and 0.015625 to seven-place accuracy, the true solutions.

y_1=y=sin□x

$$y_2 = \frac{dy}{dx} = \cos x, \qquad (3.27)$$

Are listed in Tables 1 and 2.

Results for step-sizes 0.03125 and 0.015625 (data sets 6 and 7) agree with the true values to seven figures. Results for larger step sizes are not of acceptable accuracy. The program has been run with even larger values of *h* (results not shown) as well. For *h* large enough, the solutions "blow up" in a fashion similar to that already observed for the Runge-kutta's method in Example 6.3[29-32]

Table 1: True solutions,

X Y = sinxdy/dx = cosx		
0.0	0.0000000	1.0000000
1.0	0.8414710	0.5403023
2.0	0.9092974	−0.4161468
3.0	0.1411200	−0.9899925
4.0	−0.7568025	−0.6536436
5.0	−0.9589243	0.2836622

Table 2: Calculated solutions at x=5

Step size, h$Y_1$$Y_2$		
1.0	−1.045312	0.1429169
0.5	−0.9627114	0.2857746
0.25	−0.9589062	0.2837483
0.125	−0.9589223	0.2836631
0.0625	−0.9589242	0.2836622
0.03125	−0.9589243	0.2836622
0.015625	−0.9589243	0.2836622
True values	−0.9589243	0.2836622

In view of the periodic nature of the solution functions, it is not surprising that as the step size approaches the length of the functional period, the solutions become meaningless. Clearly, for step sizes larger than the period, virtually all local information about the curvature of the function is lost; it would be unreasonable to expect accurate solutions in such cases.

CONCLUSION

The findings of the above experiment confirms that the linear multistep methods are indeed fixed point iteration methods as stated in the main result.

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