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FIXED POINT THEORY AND ITERATIVE PROCEDURES: FROM CLASSICAL THEOREMS TO MODERN DEVELOPMENTS

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ABSTRACT

The notion of fixed point theory is the key of nonlinear analysis in a specific way as it composes prominent tools to find existence and uniqueness of solutions in number of non linear problems and also provide methods to find solutions of these problems. The main idea behind the work on fixed point theory is a way to recognize conditions to be applied on the space or on the map or both simultaneously that will ensure fixed point. The paper purposes to provide a comprehensive review of classical and modern fixed point theorems and their applications to more general spaces, with a particular focus on iterative procedures. The review will cover fundamental concepts and techniques of fixed point theory, as well as recent developments in this field.

Keywords: Banach space, Cauchy Sequence, Mapping, Topological vector space

INTRODUCTION

The ramifications of fixed point theory have an extensive application in geometry, analysis, topology, differential equation, numerical analysis and integral equations and enrich the field of research in an effective manner. Since it successfully produced numerous results in various fields of mathematics, it paves a way for modern researchers to apply these results in new frames of research work. One of the most promising directions in modern fixed point theory is the study of multi-valued maps, which have various applications in economics, physics, and computer science. On the basis of implemented norms, it has been categorised in the subsequent prime zones i.e., Topological fixed point theory, Metric fixed point theory and Set theoretic fixed point theory. Sometimes many necessary or sufficient conditions for the existence of fixed points involve various mixtures of these three areas because of non metric nature of metric methods which are often useful in proving results. Iterative procedures are important in fixed point theory because they provide a constructive means of finding fixed points. In many cases, it is difficult to determine the existence of fixed points analytically, but iterative procedures can be used to approximate them numerically. Additionally, iterative procedures often provide efficient algorithms for finding fixed points, especially in cases where other methods are not feasible. Iterative procedures also have a close connection to dynamical systems, which play an important role in fixed point theory. In dynamical systems, fixed points are viewed as equilibrium points, and the iterative process is seen as a way of exploring the behavior of the system around the fixed point. The study of iterative procedures has led to the extension of classical fixed point theorems to more general spaces and the development of new methods for solving nonlinear problems. With the advance development of fixed point theory, it is progressively realised that theorems based on the concept of fixed points are the principal tool in applied mathematics also. The concept of fixed point has reference to domain points of a function that are mapped onto itself.

Definition 1(Fixed Point, [15]): *The fixed point of a function u:* $\boldsymbol{Z} \rightarrow \boldsymbol{Z}$ *is an element a of* $\boldsymbol{Z}$ *satisfying u(a) = a, where* $\boldsymbol{Z}$ *is a non empty set.*

L.E.J. Brouwer [15] contributed a theorem in 1912 that focuses on existence of fixed point(s) under suitable conditions which played a key role in research extension in this field. A number of generalizations and extensions have been recorded on the basis of this theorem. The theorem seems to be inspired by intermediate value theorem in calculus and Poincare Birkhoff theorem in dynamical system. In actual, Poincare [75] was the pioneer to introduce the concept of fixed point in 1886.

Theorem 1(Brouwer Fixed Point Theorem, [15]): *Any continuous map of a closed ball in* $\boldsymbol{R}^n$ *into itself must have a fixed point.*

The major gaps noticed in Brouwer's existence principle were the following: It doesn't provide any method to determine solution and its uniqueness. Further, it is not applicable to infinite dimensional space.

Contraction principle was the major achievement of Banach that filled former gap in Brouwer's existence principle for fixed points i.e., to find solution and its uniqueness.

Definition 2(Contraction Map, [12]): *A map u:* $\boldsymbol{Z} \rightarrow \boldsymbol{Z}$ *is said to be a contraction map, if* $d(ua, ub) \leq k\, d(a, b)$, *where* $\boldsymbol{Z}$ *is a metric space,* $a, b \in \boldsymbol{Z}$ *and* $0 \leq k < 1$.

It has already been proved in [12] that every contraction map is a continuous map, but not vice versa.

Counter example: u(a) = 2a is a continuous map but it is not a contraction map. In 1837, introduction of successive approximation by Liouville and systematically proved by Picard in 1890 gives a way to formulate Banach contraction principle [12].

In 1922, Stephen Banach mapped out the classical theorem in non - linear analysis which is stated as follow:

Theorem 2(Banach Contraction Principle, [12]): *If* $\boldsymbol{Z}$ *is a complete metric space and u:* $\boldsymbol{Z} \rightarrow \boldsymbol{Z}$ *is a contraction map, then f has a unique fixed point.*

It also underlines a method to work out the fixed point with iteration process. Iterations that can easily be operated on computers make it worth popular. The principle given by Stephen Banach also helped in emergence of other pre - eminent fixed point theorems. It has also been generalised by many researchers in various ways like Kannan [45], Chatterjee [16], Bianchini [14], Riech [76], Sehgal [83], Hardy and Rogers [36] etc. With computer based applications of fixed point theory, new trends in modern research are being invented through iteration based algorithms and coding on it.

In 1930, the latter condition in Brouwer's fixed point theorem was relaxed by Schauer by incorporating the closed bounded convex subset of infinite dimensional Banach Space. The result is known as Schauder fixed point theorem.

Theorem 3(Schauder Fixed Point Theorem, [81]): *If* $\boldsymbol{Z}$ *is a compact, convex subset of a Banach space* $\boldsymbol{A}$ *and u:* $\boldsymbol{Z} \rightarrow \boldsymbol{Z}$ *is a continuous function, then u has a fixed point.*

Compactness condition imposed on space A used in definition of Schauder's theorem has very strong impact but it's not possible to have compact setting for most of problems in analysis. Therefore, to relax compactness condition was major hindrance in extension of the theory of the concept. Schauder itself generalized his theorem as follows by relaxing the condition of compactness.

Theorem 4(Schauder Generalized Fixed Point Theorem, [81]): *If* $\boldsymbol{Z}$ *is a closed bounded convex subset of a Banach space* $\boldsymbol{A}$ *and u:* $\boldsymbol{Z} \rightarrow \boldsymbol{Z}$ *is continuous map such that u(*$\boldsymbol{Z}$*) is compact, then u has a fixed point.*

In 1935, Tychonoff generalized this existence principle to locally convex topological vector spaces [94].

Theorem 5(Tychonoff Fixed Point Theorem, [94]): *If* $\boldsymbol{Z}$ *is a nonempty compact convex subset of a locally convex topological vector space* $\boldsymbol{A}$ *and u:* $\boldsymbol{Z} \rightarrow \boldsymbol{Z}$ *is a continuous map, then f has a fixed point.*

Since the beginning of 20th century, it has been observed that a number of researchers devoted their time to investigate common fixed point for pair of mappings in contractive conditions. Several researches derived common fixed point for mapping in different spaces. Jungck [41] generalized the Banach Contraction Principle in a new way. Although it has plenty of applications but it has limitations due to continuity of mappings. These days many research articles have been published about contractive condition without any requirement of continuity of mappings.

A novel twist was noted in the development of the theory of fixed point when in 1982, Sessa [84] replaced the condition of commutativity by weak comutativity in Jugck's theorem [41]. After that, the condition of weak commutativity for mappings in a number of fixed point results was also relaxed by many researchers. Further, the new findings have widely been used by so many authors like Jungck et al. [42], Pathak and Khan [70], Al-Thagafi and Shahzad [6], Singh [88], Cho et al. [22], Pathak and Ume [71], Pathak et. al. [65-69] etc. b-metric space, partial metric space, generalized metric space, modular metric space, fuzzy metric space are some further categorized fields of metric space where fixed point theorems have been investigated and applied.

Bakhtin [11], in 1989, presented the concept of b-metric space which is a generalized version of classical metric space and used by Czerwik [26] to extend the fixed point theories in 1993.

Definition 3(b – Metric Space, [2])*: Let Z be a non-empty set and let $s \geq 1$ be a given real number. A function $k: Z \times Z \to [0, \infty)$ is said to be a b-metric if and only if for all a, b, c following conditions are satisfied:*

- ***(i)*** *$k(a, b) = 0$ if and only if $a = b$;*
- ***(ii)*** *$k(a, b) = k(b, a)$ for all $a, b \in Z$;*
- ***(iii)*** *$k(a, b) \leq s[k(a, c) + k(c, b)]$ for all $a, b, c \in Z$.*

Then (Z, k, s) is called a b-metric space.

Followed by Czerwik, several researchers provided important fixed point results in b-metric senseusing different approaches in case of single valued mappings or multivalued mappings or both simultaneously. (For reference see [9], [21], [28], [40], [48], [64], [72]). Czerwik [26-27] proved that a classical metric space is always a b-metric space but not conversely. Singh and Prasad [87] justified with examples that b-metric spaces need not to be a metric space.
In 1994, Matthews [53] submitted the concept of non-zero self distant metric called partial metric and it validated the Banach Contraction Principle, the mile stone of fixed point theory, in context of partial metric space [53].

Definition4(Partial Metric Space, [53]): *A partial metric on a non empty set* ***Z*** *is a function $q: \mathbf{Z} \times \mathbf{Z} \to [0,\infty)$ such that for all $a, b, c \in \mathbf{Z}$*
(i) $a = b \Longleftrightarrow q(a, a) = q(a, b) = q(b, b)$;
(ii) $q(a, a) \leq q(a, b)$;
(iii) $q(a, b) = q(b, a)$;
(iv) $q(a, b) \leq q(a, c) + q(c, b) - q(c, c)$.
A partial metric space is a pair (Z, q) such that ***Z*** *is a nonempty set and q is a partial metric on Z.*
It is to be noted that if q(a,b) =0, then from axiom (i) and (ii), a = b, but not vice versa I.e., if a = b, then q(a, b) need not to be zero**.** As an example, one can consider the pair $([0, \infty), q)$, where $q(a, b) = \max. \{a, b\}$ for all $a, b \in [0,\infty)$.

Definition 5(Completeness in Partial Metric Space, [53]): *Let (Z, q) be a partial metric space.*
(i) A sequence $\{a_n\}$ in Z is said to be a Cauchy sequence if, $\lim_{(m, n \to +\infty)} q(a_n, a_m)$ exists and is finite.
(ii) (Z, q) is said to be complete if every Cauchy sequence $\{a_n\}$ in Z converges with respect to τ_q to a point $a \in Z$ such that $\lim_{(m, n \to +\infty)} q(a_n, a) = q(a, a)$. In this case, we say that the partial metric q is complete.

Rush, in 2008, discussed existence and uniqueness of fixed points, methods to find fixed points, the convergence of successive approximation to find fixed points of many non- linear problems of metric fixed point theory in context of partial metric spaces. Inspired by Rush, Bessem Samel et al. proved several new fixed point theorems on complete metric spaces and also discussed Matthews' fixed point theorem, Banach fixed point theorem, Kannan's fixed point theorem, Reich's fixed point theorem and Chatterjee's fixed point theorem on partial metric spaces as a particular case of results proved by them. Aydi et al. [8] presented the notion of partial Hausdorff metric in 2012 and generalized the most useful result of multivalued analysis, that is, "Nadler's multivalued contraction theorem" in complete partial metric space. A number of researchers have also demonstrated the results of fixed points based on specific type of mappings in partial metric settings.

L.A. Zadeh [99], in 1965, was the pioneer to introduce the notion of fuzzy set in Fuzzy mathematics. In classical set theory, mathematicians deal with well defined collection of objects. Later on, the vague collection of objects was realised as an important concept to construct new frames of set theory as the idea behind the foundation of term fuzzy is vagueness in daily life. For example, collection of best Indian cricketers can't be studied in classical set theory but fuzzy set theory enables one to study such type of problems. In fuzzy sense, a degree of membership is designed for each object which is to be introduced in a particular fuzzy set on basis of some pre-determined norms. The above explained example forms a fuzzy set as degree of membership assigned to each object enables one to explain an object belongs to the set with a precise degree of membership. The fuzzy point of view of set theory provides researchers a new framework to work with this notion. In actual, it's a theory based on degree or elasticity of an object with an existence of pre-defined norms. The fundamental work produced by Zadeh leads to progressive work made in field of fuzzy mathematics and its application to all other branches of the subject.

The idea of fuzzy metric space presented by Kramosil and Michalek (1975) [50] paves a way for the introduction of fixed point theory in fuzzy metric spaces. M. Grabiec, with the idea of completeness axiom in fuzzy metric spaces [Referred as G - Complete Metric Space], extended Banach contraction principle. Some fixed point theorems for contraction type mappings in G - complete metric space were also proved by Fang [31] by following M. Grabiec's work. George and Virmani [32], in 1994, modified Grabiec's definition of Cauchy sequence as the set of real numbers is not order complete in Grabiec's completeness axiom sense.

Definition 6(Continuous t – norm, [82]): *A binary operation #: $[0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-norm if it satisfies the following conditions:*

(i) # is commutative and associative;
(ii) # is continuous;
(iii) $a \# 1 = a$ for every $a \in [0, 1]$;
(iv) $a \# b \leq c \# d$ whenever $a \leq c$, $b \leq d$ and $a, b, c, d \in [0, 1]$.

Definition 7 (Fuzzy Metric Space, [32]): *A fuzzy metric space is an ordered triple ($\mathbf{Z}$, M, #) such that $\mathbf{Z}$ is a nonempty set, # is a continuous t-norm and M is a fuzzy set on $\mathbf{Z} \times \mathbf{Z} \times (0, \infty)$ satisfying the following conditions, for all $a, b, c \in \mathbf{Z}$, $s, t > 0$:*
(i) $M(a, b, c) > 0$;
(ii) $M(a, b, t) = 1$ if and only if $a = b$;
(iii) $M(a, b, t) = M(b, a, t)$;
(iv) $M(a, b, t) \# M(b, c, s) \leq M(a, c, t + s)$;
(v) $M(a, b, .) : (0,\infty) \to (0, 1]$ is continuous.

Definition 8(Completeness in Fuzzy Metric Space [32], [34]): *Let ($\mathbf{Z}$, M, #) be a fuzzy metric space. Then:*
(i) A sequence $\{a_n\}$ is said to converge to a in $\mathbf{Z}$, denoted by $a_n \to a$, if and only if $\lim_{(n\to\infty)} M(a_n, a, t) = 1$ for all $t > 0$;
i.e. for each $r \in (0, 1)$ and $t > 0$, there exists $n0 \in N$ such that $M(a_n, a, t) > 1 - r$ for all $n \geq n0$.
(ii) A sequence $\{a_n\}$ in $\mathbf{Z}$ is an M-Cauchy sequence if and only if for each $\epsilon \in (0, 1)$, $t > 0$, there exists $n0 \in N$ such that $M(a_m, a_n, t) > 1 - \epsilon$ for any $m, n \geq n0$;
(iii) The fuzzy metric space ($\mathbf{Z}$, M, #) is called M-complete if every M-Cauchy sequence is convergent.

They also introduced the notion of completeness in fuzzy metric space [Referred as M - Complete Metric Space] and some fixed point theorems have also been proved on the basis of this concept by different researchers. On basis of two kinds of complete metric space, a vast literature on fixed point theory has been constructed and studied.
In 1992, Dhage [29] initialized a new version of classical metric space which is known as generalized metric space [D-metric space]. He contributed fundamental properties of D-metric spaces in topological sense. Mustafa and Sims [57], in 2006, established a new generalized metric space by removing discrepancies in Dhage's work. It is called G-metric space which is the more appropriate concept of generalized metric space. Abbas and Rhoades [2] worked on common fixed point problems in G-metric space. Following Abbas and Rhoades, many investigators established common fixed point

results for self mappings satisfying various contractive conditions in G-metric spaces [9, 10, 22, 23, 24, 33, 43, 55-59, 85, 86]. Kaewcharoen and Kaewkhao [44] and Nedal et al. [61] proved fixed point results for single-valued and multivalued mappings in Hausdorff G-distance sense. In what follows, one can recall some preliminaries of G-metric space. The definition of G-metric space given by Mustafa and Sims [57]:

Definition 9*(G – Metric Space, [*57]): *Let **Z** be a non-empty set and K: **Z** × **Z** × **Z** → $\boldsymbol{R}^+$ be a function satisfying the following properties:*
(i) K (a, b, c) = 0 if a = b = c;
(ii) 0 < K (a, a, b), for all a, b ∈Z, with a ≠ b;
(iii) K (a, b, b) ≤ K (a, b, c), for all a, b, c ∈ Z, with c ≠ b;
(iv) K (a, b, c) = K (a, c, b) = K (b, c, a) = ... (Symmetric in all three variables);
(v) K (a, b, c) ≤ K (a, r, r) + K (r, b, c) for all a, b, c, r ∈ Z (Rectangle inequality)

*Then the function K is called a generalized metric or, more specifically, a G-metric on **Z** and the pair (**Z**, K) is called a G-metric space.*

Definition 10(Symmetric G – Metric Space, [57]): *A G-metric space is said to be symmetric if K (a, b, b) = K (b, a, a) for all a, b ∈**Z**.*

Example 1 [57]: *Let (**Z**, d) be a usual metric space. Then the function K: **Z** × **Z** × **Z** → $\boldsymbol{R}^+$ defined by K (a, b, c) = max{d(a, b), d(b, c), d(c, a)} or K(a, b, c) = d(a, b) + d(b, c) + d(c, a) for all a, b, c∈**Z** is a G-metric space.*

Definition 11(G –Cauchy Sequence, [57]): *Let (**Z**, K) be a G-metric space. Then a sequence $\{a_n\}$ in **Z** is:*
(i) a G- convergent sequence if for any 0, there exist a∈Z and N∈ℕ such that $K(a, a_n, a_m) < \epsilon$, for all n, m ≥ N;
(ii) A G-Cauchy sequence if for any ϵ is positive, there exist N ∈ℕ such that $K(a_n, a_m, a_l) < \epsilon$, for all n, m, l ≥ N.

Proposition 1 [57]: *Let (**Z**, K) be a G-metric space and $\{a_n\}$ be a sequence in **Z**. Then the following are equivalent:*
(i) $\{a_n\}$ converges to a;
(ii) $K (a_n, a_n, a) \to 0$ as $n \to \infty$;
(iii) $K (a_n, a, a) \to 0$ as $n \to \infty$;
(ib) $K (a_m, a_n, a) \to 0$ as $m, n \to \infty$.

Chistyakov [18-20] introduced the concept of modular metric spaces by including the time parameter as third argument. Nakano's [60] non linear notion of linear modular spaces and Musielack and Orliez's [54] idea of modular function spaces are important illustrations of these modular metric spaces. There are numerous applications of modular metric spaces but in 2014, Abdou and Khamsi [3] introduced the results of fixed point theorems for the first time in sense of modular metric spaces and it opened the new directions in fixed point theory. After this breakthrough, fixed point theory on modular Matrix spaces was enormously explained by so many researchers.
As discussed earlier, the fundamental approach of fixed point theory is to find the suitable conditions to be imposed either on the mapping or on the set where the mapping is well defined in order to get the fixed point solution of non-linear problems. Once these conditions are fulfilled, a large number of results based on fixed point theorems can be obtained for different type of mappings with the help of iteration procedure. An iteration procedure is an algorithm based scheme where the same steps are repeated to get the required result. Each repetition in this process is known as iteration and the result obtained from each former iteration is used as input for the next iteration. The basic idea behind the iteration process is that it depends on picking up some initial point on the space that would lead a sequence converging to the fixed point solution of a specific non linear problem.
The well known method of successive iteration procedure formulated by Picard [74] is defined by the sequence $\{r_n\}$ as follows:
$r_{n+1} = \mathbf{K}r_n$, for all $n \in N$.
This iteration procedure converges/leads to the fixed point of contractive type mappings.

Mann [52] observed that iteration procedure stated by Picard may fail to converge in sense of non - expansive mappings. It was also claimed by him that iteration procedure developed by Picard may not be applicable for other class of non - linear mappings. To deal with this discrepancy, Mann [52], in 1953, detected a new iteration procedure defined by the sequence $\{k_n\} \in (0, 1)$ as follows:
$r_{n+1} = (1 - k_n)\, r_n + k_n\mathbf{K}r_n$ for all $n \in N$.

Krasnoselskii [51], in 1955, founded that iteration procedure constituted by Picard fails to converge to fixed point in case of non - expansive mapping **K** even the mapping **K** assure unique fixed point.

He presented an improved version of the iteration procedure with the constant $k \in (0, 1)$ as follows:

$r_{n+1} = (1 - k)\, r_n + k\mathbf{K}r_n$ for all $n \in N$.

The iteration procedure detected by Mann and Krasnoselskii are identical in case of the sequence $\{k_n\}$ be a constant sequence. An important drawback noted in Mann's iteration procedure is that it is not applicable in case of pseudo contractive mapping. To remove this drawback, Ishikawa [39], presented an improved version of iteration procedure given by Mann and Krasnoselskii to approximate the fixed points in sense of pseudo contractive mappings by the iterative scheme:

$r_{n+1} = (1 - k_n)\, r_n + k_n\mathbf{K}s_n$

$s_n = (1 - l_n)\, r_n + l_n\mathbf{K}r_n$

for all $n \in N$, where $\{k_n\}$ and $\{l_n\}$ are the sequences in (0, 1).

Chidume and Mutangadura [17] noted that iteration procedure developed by Mann is not applicable in finding fixed point of lipschitzian pseudo contractive mappings whereas Ishikawa iteration procedure is applicable. Rhoades and Solutz [78] even observed that iteration procedure developed by Mann and Krasnoselskii are equivalent for distinct type of nonlinear mappings. Rhoades [77] also noted the rate of convergence of iteration procedures developed by Mann and Ishikawa. He remarked that iteration procedure developed by Mann has faster rate of convergence as compared to Ishikawa's iteration procedure in case of decreasing functions. He claimed that iteration procedure developed by Ishikawa is better than Mann's iteration procedure in case of increasing functions. He also added that Mann's iteration procedure appears to be independent of choosing initial point.

Noor [62], in 2000, developed a three - step iteration procedure as follows:

$r_{n+1} = (1 - k_n)r_n + k_n\mathbf{K}s_n$

$s_n = (1 - l_n)r_n + l_n\mathbf{K}t_n$

$t_n = (1 - m_n)r_n + m_n\mathbf{K}r_n$

for all $n \in N$, where $\{k_n\}$, $\{l_n\}$ and $\{m_n\}$are the sequences in (0, 1).

Agrawal et al. [5], in 2007, presented a two - step iteration procedure to approximate fixed points of asymptotically non - expansive mappings as follows:

$r_{n+1} = (1 - k_n)\mathbf{K}r_n + k_n\mathbf{K}s_n$

$s_n = (1 - l_n)r_n + l_n\mathbf{K}r_n$

for all $n \in N$, where $\{k_n\}$and $\{l_n\}$ are the sequences in (0, 1).

Agrawal et al. [5] established that iteration procedure developed by them approximate fixed point of a contraction mapping with equal rate of convergence as Picard iteration procedure does. It approximate fixed points faster than Mann's iteration procedure. They also claimed that iteration procedure developed by them is independent of Mann and Ishikawa iteration procedures. Hussain et al. [38] stated that the iteration procedure given by Agarwal et al. approximate fixed points with higher rate of convergence as compared to Mann and Ishikawa iteration procedure in sense of quasi - contractive operators. The iteration procedure defined by Agrawal et al. [5] has an equivalence with Ishikawa's procedure of iteration. It improves the rate of convergence for finding fixed points of nearly asymptotically non - expansive mappings.

Soltuz [89], in 2007, noted that the iteration procedures developed by Mann, Ishikawa and Noor are equivalent in sense of quasi contractive mappings in a Banach Space. Phuengratta and Suantai [73] presented a new iteration procedure in 2011 which is known as SP - iteration procedure. It follows as:

$t_n = m_n\mathbf{K}r_n + (1 - m_n)r_n$

$s_n = l_n\mathbf{K}t_n + (1 - l_n)t_n$

$r_{n+1} = k_n\mathbf{K}s_n + (1 - k_n)s_n$

for all $n \in N$, where $\{k_n\}$, $\{l_n\}$ and $\{m_n\}$are the sequences in (0, 1).

Chugh et al. [25], in 2012, presented the notion of CR- iteration procedure as follows:

$r_{n+1} = (1 - k_n)s_n + k_n\mathbf{K}s_n$

$s_n = (1 - l_n)\mathbf{K}r_n + l_n\mathbf{K}t_n$

$t_n = (1 - m_n)r_n + m_n\mathbf{K}r_n$

for all $n \in N$, where $\{k_n\}$, $\{l_n\}$ and $\{m_n\}$ are the sequences in (0, 1).

He observed that CR- iteration procedure has faster rate of convergence than some other existing iteration procedures developed for quasi contractive operators.

Sahu [79] and Khan [49] independently developed the idea of Normal - S iteration procedure/ Picard - Mann hybrid procedure as follows:

$r_{n+1} = \mathbf{K}s_n$

$s_n = (1 - k_n)r_n + k_n\mathbf{K}r_n$

for all $n \in N$, where $\{k_n\}$ is a sequences in (0, 1). Investigations made in [49] affirmed that rate to approximate fixed points by normal S iteration procedure is as fast as Picard iteration procedure. It is also faster than some other iteration procedures developed by Mann, Ishikawa, Noor etc.

After this, Okeke and Abbas [63] introduced Picard - Krasnoselskii hybrid iterative process as follows:
$r_{n+1} = \mathbf{K}s_n$
$s_n = (1 - k)r_n + k\mathbf{K}r_n$
where $r_0 \in H$ and $k \in (0,1)$. In addition, Okeke and Abbas [63] affirmed the faster rate of convergence to approximate fixed points by the iteration procedure developed by him as compared to Picard, Ishikawa, Mann and Krasnoselskii iteration procedure for quasi contractive operators in sense of Berinde [13].

Gursoy and Karakaya [35] added a new chapter in literature of iteration procedures by developing Picard - S - iteration procedure as follows:
$t_n = (1 - l_n)r_n + l_n\mathbf{K}r_n$
$s_n = (1 - k_n)r_n + k_n\mathbf{K}t_n$
$r_{n+1} = \mathbf{K}s_n$,
for all $n \in N$, where $\{k_n\}$and $\{l_n\}$ are the sequences in (0, 1). Gursoy and Karakaya [35] noted that iteration procedure, to approximate fixed points, developed by them has higher rate of convergence than Normal - S - iteration procedure, CR iteration procedure, SP iteration procedure and other iterative schemes developed by Ishikawa, Noor, Mann, Picard etc.

Abbas and Nazir [1], in 2014, expanded a new three - step iteration procedure for fixing fixed points of non - linear problems in sense of uniformly convex Banach space as follows:
$t_n = (1 - m)\ r_n + m_n\mathbf{K}r_n$
$s_n = (1 - l_n)\ \mathbf{K}s_n + l_n\mathbf{K}t_n$
$r_{n+1} = (1 - k)\ \mathbf{K}t_n + k_n\mathbf{K}t_n$
for all $n \in N$ and $\{k_n\}$, $\{l_n\}$ and $\{m_n\}$ are sequences in (0, 1). Investigations made in [1] affirmed that iteration procedure, to find fixed point of contraction mappings, developed in it has higher rate of convergence than all of Agarwal, Noor, Ishikawa, Mann and Picard iteration procedure.

Thakur et al. [91], in 2014, established an iteration procedure as follows:
$t_n = (1 - m_n)r_n + m_n\mathbf{K}r_n$
$s_n = (1 - l_n)t_n + l_n\mathbf{K}t_n$
$r_{n+1} = (1 - k_n)\mathbf{K}r_n + k_n\mathbf{K}s_n$
for all $n \in N$ and $\{k_n\}$, $\{l_n\}$ and $\{m_n\}$ are sequences in (0, 1).

Karakaya et al. [47] set up a two - step iteration procedure by following the research of Gursoy and Karakaya [35] in 2015. It is named as Vatan two - step iteration procedure. It follows as:
$s_n = \mathbf{K}((1 - k_n)r_n + k_n\mathbf{K}r_n)$
$r_{n+1} = \mathbf{K}((1 - l_n)s_n + l_n\mathbf{K}s_n)$
for all $n \in N$ and $\{k_n\}$ and $\{l_n\}$ are the sequences in (0, 1).

Thakur et al. [91] developed a new iteration procedure in 2016. It follows as:
$t_n = (1 - l_n)r_n + l_n\mathbf{K}r_n$,
$s_n = \mathbf{K}((1 - k_n)r_n + k_nt_n)$,
$r_{n+1} = \mathbf{K}s_n$,
for all $n \in N$ and $\{k_n\}$ and $\{l_n\}$are the sequences in (0, 1).

In 2016, Ullah and Arshad [95] established AK - iteration procedure which is defined as follows:
$t_n = \mathbf{K}((1 - l_n)r_n + l_n\mathbf{K}r_n)$
$s_n = \mathbf{K}((1 - k_n)t_n + k_n\mathbf{K}t_n)$
$r_{n+1} = \mathbf{K}s_n$
for all $n \in N$ and $\{a_n\}$ and $\{b_n\}$are the sequences in (0, 1). Ullah and Arshad [95] also claimed that iteration procedure detected by them has higher rate of convergence over iteration procedure developed by Thakur et al.

Sahu et al. [80] and Thakur et al. [92-93], in 2016, developed the same iteration procedure for finding fixed points of non expansive mapping in sense of uniformly convex Banach space as follows:
$t_n = (1 - m_n)r_n + m_n\mathbf{K}r_n$
$s_n = (1 - l_n)t_n + l_n\mathbf{K}t_n$
$r_{n+1} = (1 - k_n)\mathbf{K}t_n + k_n\mathbf{K}s_n$
for all $n \in N$ and $\{k_n\}$, $\{l_n\}$ and $\{m_n\}$are sequences in (0, 1).

Karakaya et al. [46], in 2017, derived an iteration procedure as follows:
$t_n = \mathbf{K}r_n$,
$s_n = (1 - k_n)t_n + k_n\mathbf{K}t_n$
$r_{n+1} = \mathbf{K}s_n$,
for all $n \in N$ and $\{k_n\} \in (0, 1)$.

Ullah and Arshad [96], in 2018, developed an iteration procedure which is known as M iteration procedure. It follows as:
$t_n = (1 - l)\ r_n + l_n\mathbf{K}r_n$,
$s_n = \mathbf{K}((1 - k_n)\ r_n + k_n\mathbf{K}t_n)$,
$r_{n+1} = \mathbf{K}s_n$,

for all $n \in N$ and $\{k_n\}$ and $\{l_n\}$ are the sequences in (0, 1).

Inspired by the investigations made by Karakaya et al. [46] and Gursoy and Kharkaya [35], Dogan and Kharkaya [30] established the following iteration procedure:

$t_n = \mathbf{K}r_n,$

$s_n = (1 - l_n)\,\mathbf{K}r_n + l_n\mathbf{K}t_n,$

$r_{n+1} = (1 - k_n)\,\mathbf{K}t_n + k_n\mathbf{K}s_n,$

for all $n \in N$, $\{k_n\}$ and $\{l_n\}$ are the sequences in (0, 1).

Investigators in [98], in 2018, established a new iteration procedure i.e., K - iteration procedure and presented a number of results based on approximation of fixed points in sense of uniformly convex Banach spaces.

$t_n = (1 - l_n)r_n + l_n\mathbf{K}r_n$

$s_n = \mathbf{K}((1 - k_n)\mathbf{K}r_n + k_n\mathbf{K}t_n)$

$r_{n+1} = \mathbf{K}s_n$

for all $n \in N$ and $\{k_n\}$ and $\{l_n\}$ are the sequences in (0, 1). This iteration procedure possesses faster rate of convergence over Picard - S - iteration and iteration procedure developed by Thakur et al.. Researchers in [97] stated some results of convergence in sense of Suzuki generalised non - expansive mappings in Banach spaces.

Ullah and Arshad [96], again in 2018, developed K - iteration procedure as follows:

$t_n = (1 - l_n)r_n + l_n\mathbf{K}r_n$

$s_n = \mathbf{K}((1 - k_n)t_n + k_n\mathbf{K}t_n)$

$r_{n+1} = \mathbf{K}s_n$

for all $n \in N$ and $\{k_n\}$ and $\{l_n\}$ are the sequences in (0, 1). Investigations made in [95] affirmed some results of weak and strong convergence in sense of Suzuki generalised non - expansive mappings via K - iteration procedure.

Atalan [7] introduced a new iteration procedure and assured better rate of convergence over some other iteration procedures in the literature.

$t_n = \mathbf{K}(\mathbf{K}(r_n))$

$s_n = \mathbf{K}((1 - k_n)kt_n + (1 - (1 - k_n)k)\mathbf{K}t_n)$

$r_{n+1} = \mathbf{K}s_n$

for all $n \in N$and $\{k_n\}$ is a sequences in (0, 1). He claimed that iteration procedure developed by him can be used to find fixed points in reference of contraction mappings. In addition, he set up an equivalence relation between M iteration procedure and iteration procedure developed by him. He assured higher rate of convergence in iteration procedure developed by him over M - iteration procedure. He also presented a number of numerical examples to demonstrate the efficiency of iteration procedure developed by him.

Following AK - iteration procedure, Hudson et al. [37], in 2019, introduced the new Picard - S - AK hybrid iteration procedure as follows:

$t_n = (1 - l_n)r_n + l_n\mathbf{K}r_n$

$s_n = \mathbf{K}((1 - k_n)\mathbf{K}r_n + k_n\mathbf{K}t_n)$

$r_{n+1} = \mathbf{K}s_n,$

for all $n \in N$ and $\{k_n\}$ and $\{l_n\}$are the sequences in (0, 1).

In 2021, Srivastava [90] introduced a new type of hybrid iterative scheme from Picard and S - iteration (Picars- hybrid iterative scheme). The sequence $\{r_n\}$ of the scheme is defined as

$r_1 = r \in S,$

$r_{n+1} = \boldsymbol{K}s_n \quad n \in I^+$

$s_n = (1 - k_n)\,\mathbf{K}r_n + k_n\mathbf{K}t_n,$

$t_n = (1 - l_n)r_n + l_n\mathbf{K}r_n\,,$

Where $\{k_n\}$ and $\{l_n\}$ are the sequences in (0, 1).

Recently, Anku et al. [4] introduced a new one parameter class of one-step fixed point iteration procedure and assured better rate of convergence over some other one-step iteration procedures in the literature as follows:

$$r_{n+1} = \frac{(k+1)r_n\mathbf{K}r_n}{kr_n + \mathbf{K}r_n} \quad k \geq 1, n \geq 0$$

The research was extended to two-step iterative scheme for $r_0 \neq 0$, given in domain D, the sequence $\{r_n\}$ in domain D is given by

$$r_{n+1} = \boldsymbol{K}\left(\frac{\mathbf{K}r_n + s_n}{2}\right)$$

$$s_n = \frac{(k+1)r_n\mathbf{K}r_n}{kr_n + \mathbf{K}r_n} \quad k \geq 1, n \geq 0$$

The research was further extended to three-step iterative scheme for $r_0 \neq 0$, given in domain D, the sequence $\{r_n\}$ in domain D is given by

$$r_{n+1} = \boldsymbol{K}s_n$$

$$s_n = \boldsymbol{K}t_n$$

$$t_n = \frac{(k+1)r_n\mathbf{K}r_n}{kr_n + \mathbf{K}r_n} \quad k \geq 0, n \geq 0$$

They also discussed the stability, strong convergence and fastness of the proposed methods and performed numerical experiments to check the applicability of the new methods.

CONCLUSION

One of the most significant developments in fixed point theory is the extension of classical fixed point theorems to more general spaces, such as non-compact spaces and non-metric spaces. Another important advancement is the study of fixed point theorems in dynamic systems, where the fixed point is viewed as an equilibrium state. Additionally, researchers have developed various applications of fixed point theory in other fields of mathematics, such as optimization, game theory and economics. The developments in fixed point theory have led to a deeper understanding of the fundamental concepts of mathematics and have opened up new avenues for research and application in various fields. These advancements are expected to continue to play a crucial role in the development of mathematics and its ramifications in the years to come.

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