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RADII FOR CERTAIN CLASS DEFINED BY DIFFERENTIAL OPERATOR

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ABSTRACT

Use of differential operators in geometric function theory has been become a topic for a lot of investigation in recent years. These investigations considered to be important because the generalized many results studied by various research. One of the most beautiful classical problems in geometric function theory is radius of star likeness and radius of convexity. In this work, we determined radius of star likeness and convexity for the class $S^{*}_{\lambda,\alpha}(\mu)$ defined by generalized differential operator $RD^{n}_{\lambda,\alpha} f(z)$. On the other hand, we used the computer software like (Wolfram Alfa Program– Complex Tool Program) for graph some special cases.

Keywords: Univalent function, radius of star like and radius of convexity.

INTRODUCTION

Let T denote the class of function $f(z)$ defined by

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad (a_k \geq 0, k \in N = \{1.2.\cdots\}) \tag{1.1}$$

Which are univalent and analytic in the open unite disc $U = \{z : |z| < 1\}$.

Definition 1.1: The class of starlike function of order μ denote by $S^{*}(\mu)$ if $f(z) \in T$ and satisfies the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \mu, \quad (0 \leq \mu < 1;\ z \in U) \tag{1.2}$$

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Definition 1.2 : The class of convex function of order μ denote by $C(\mu)$ if $f(z) \in T$ and satisfies the condition

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \mu, \qquad (0 \leq \mu < 1,\ z \in U). \tag{1.3}$$

Note that $S^*(0)\equiv S^*$ is the class of starlike functions and $C(0)\equiv C$ is the class of convex functions. Ruscheweyh [5] defined the differential operator $R^n f(z)$ as follows

$$R^0 f(z)=f(z),$$
$$R^1 f(z)=z f'(z)$$
$$(\delta+1)R^{\delta+1} f(z)=z\left(R^{\delta} f(z)\right)'+nR^{\delta} f(z)=z+\sum_{k=2}^{\infty}\delta(n,K)\,a_k z^k, \quad (1.4)$$

where $n\in N_0=N\cup\{0\}, z\in U$ and

$$\delta(n,K)=\frac{(n+k-1)!}{n!\,(k-1)!}. \quad (1.5)$$

Al-Oboudi [2] defined the differential operator $D_\lambda^n f(z)$ by:

$$D_\lambda^0 f(z)=f(z),$$
$$D_\lambda f(z)=D_\lambda f(z)=(1-\lambda)f(z)+\lambda z f'(z)$$
$$D_\lambda^n f(z)=D_\lambda\left(D_\lambda^{n-1} f(z)\right)=z-\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n a_k z^k, \quad (n\in N_0, z\in U), \quad (1.6)$$

Lupas [3] defined the generalized differential operator $RD_{\lambda,\alpha}^n f(z)$ as linear combination of Ruscheweyh operator and Al-Oboudi differential operator by:

$$RD_{\lambda,\alpha}^n f(z)=(1-\alpha)R^n f(z)+\alpha D_\lambda^n f(z). \quad (1.7)$$

By simple calculate, we have

$$RD_{\lambda,\alpha}^n f(z)=z-\sum_{k=2}^{\infty}\left[\alpha\left[1+(k-1)\lambda\right]^n+(1-\alpha)\delta(n,k)\right]a_k z^k. \quad (1.8)$$

From equation (1.7), we note that

$$RD_{\lambda,n}^1 f(z)=z f'(z) \quad (1.9)$$

Now, by taking different value of the parameters n,λ,α and μ, we get some special cases of the operator $RD_{\lambda,\alpha}^n f(z)$, for example.

i. $RD_{\lambda,1}^n f(z)=D_\lambda^n f(z)$ studied by Al-Oboudi [2];
ii. $RD_{1,1}^n=S^n f(z)$, studied by Sâlâgean [6];
iii. $RD_{\lambda,0}^n f(z)=R^n f(z)$, studied by Ruscheweyh [5].

In 2014, Lupas and Andrei [4] use the generalized differential operator $RD_{\lambda,\alpha}^n f(z)$ to define the class $S_{\lambda,\alpha}^n(\mu)$, which consists of all function $f(z)\in T$ satisfies the condition

$$\operatorname{Re}\left\{\frac{z\left(D_{\lambda,\alpha}^n f(z)\right)'}{D_{\lambda,\alpha}^n f(z)}\right\}>\mu, \quad (z\in U; 0\le\mu<1), \quad (1.10)$$

where $RD_{\lambda,\alpha}^n f(z)$ given by (1.8).

By specializing the parameters n,λ,α and μ, in the definition of the class $S_{\lambda,\alpha}^n(\mu)$ can be reduced know classes :

i. $S^{0}_{\lambda,\alpha}(\mu) = T^{*}(\mu)$ studied by Silverman [7];

ii. $S^{1}_{\lambda,\alpha}(\mu) = C(\mu)$ studied by Silverman [7];

iii. $S^{n}_{\lambda,0}(\mu) = S^{*}_{\lambda}(\mu)$ studied by Ahuja [1];

iv. Put $\alpha = 0$, we get the class defined as follows:

v. $$\operatorname{Re}\left\{\frac{z\left(D^{n}_{\lambda} f(z)\right)'}{D^{n}_{\lambda} f(z)}\right\} > \mu, \quad \left(n \in N_0,\ z \in U; 0 \le \mu < 1\right);$$

vi. Put $\alpha = 1$ and $\lambda = 1$, we get the class defined as follows

vii. $$\operatorname{Re}\left\{\frac{z\left(S^{n} f(z)\right)'}{S^{n} f(z)}\right\} > \mu, \quad \left(z \in U; 0 \le \mu < 1\right).$$

The problem of coefficient estimates is one of interesting problems which was studied by researchers for certain classes in the open unit disc. Closely related to this problem Using the results of Lupas and Andrei [4] to determine radius of star likeness and radius of convexity details with some application of computers software .

RADII OF STARLIKENESS AND CONVEXITY

In order to prove our results, we need the following Lemma due to Lupas and Andrei [4] :

Lemma 2.1:

Let the function $f(z)$ defined by (1.1) belong to the class T, if

$$\sum_{k=2}^{\infty}(k-\mu)\left[\alpha\left[1+(k-1)\lambda\right]^{n} + (1-\alpha)\delta(n,k)\right]a_k \le 1-\mu, \tag{2.1}$$

then $f(z) \in S^{n}_{\lambda,\alpha}(\mu)$, where $0 \le \mu < 1$ and $\delta(n,k)$ defined by (1.5). The result is sharp for the function

$$f(z) = z - \frac{1-\mu}{(k-\mu)\left[\alpha\left[1+(k-1)\lambda\right]^{n} + (1-\alpha)\delta(n,k)\right]} z^{k}, \quad (k \ge 2). \tag{2. 2}$$

Now we study radius of starlikeness for the function $f(z) \in T$ belong to the classes $S^{n}_{\lambda,\alpha}(\mu)$ by obtaining the coefficient estimates.

Theorem 2.1:

Let the function $f(z)$ given by (1.1) be in the class $S^{n}_{\lambda,\alpha}(\mu)$,then $f(z)$ is starlike of order $\rho(0 \le \rho < 1)$ in $|z| < r_1(n.,k,\lambda,\alpha,\mu,\rho)$, where

$$r_1(n,k,\lambda,\alpha,\mu,\rho) = \inf_{k \ge 2}\left[\frac{(1-\rho)(k-\mu)\left[\alpha\left[1+(k-1)\lambda\right]^{n} + (1-\alpha)\delta(n,k)\right]}{(k-\rho)(1-\mu)}\right]^{\frac{1}{k-1}}. \tag{2.3}$$

The result is sharp for the function $f(z)$ defined by (2.2).

Proof

To find the radius of starlike of order α, it sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1-\rho. \tag{2.4}$$

By simple calculations, we get

$$\left|\frac{zf'(z)}{f(z)}-1\right|=\left|\frac{zf'(z)-f(z)}{f(z)}\right|<\frac{\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty}a_k|z|^{k-1}}. \tag{2.5}$$

Thus equation (2.4) satisfies if

$$\sum_{k=2}^{\infty}\left(\frac{k-\rho}{1-\rho}\right)a_k|z|^{k-1}<1. \tag{2.6}$$

Since $f(z)\in S_{\lambda,\alpha}^{n}(\mu)$, Lemma 2.1 conforms that

$$\sum_{k=2}^{\infty}\frac{(k-\mu)\left[\alpha\left[1+(k-1)\lambda\right]^{n}+(1-\alpha)\delta(n,k)\right]}{1-\mu}a_k\leq 1,$$
(2.7)

hence, from (2.6) and (2.7), we have

$$\left(\frac{k-\rho}{1-\rho}\right)|z|^{k-1}<\frac{(k-\mu)\left[\alpha\left[1+(k-1)\lambda\right]^{n}+(1-\alpha)\delta(n,k)\right]}{1-\mu}. \tag{2.8}$$

Solving (2.8) for $|z|$, we get

$$|z|<\left[\frac{(1-\rho)(k-\mu)\left[\alpha\left[1+(k-1)\lambda\right]^{n}+(1-\alpha)\delta(n,k)\right]}{(1-\mu)(k-\rho)}\right]^{\frac{1}{k-1}}. \tag{2.9}$$

Thus, the proof of Theorem 2.1 is completed.
Put $n=0$ in Theorem 2.1, we get the following corollary

Corollary 2.1:
Let the function $f(z)$ defined by (1.1) be in the class $T^{*}(\mu)$, Then $f(z)$ is starlike in $|z|<r_2(k,\rho,\mu)$, where

$$r_2(k,\rho,\mu)=\inf_{k\geq 2}\left[\frac{(1-\rho)(k-\mu)}{(k-\rho)(1-\mu)}\right]^{\frac{1}{k-1}},\quad (k\geq 2). \tag{2.10}$$

The result is sharp for the function

$$f(z)=z-\frac{1-\mu}{(k-\mu)}z^{k},\quad (k\geq 2). \tag{2.11}$$

Put $k=3$ in Corollary 2.1, we get

Example 2.1: Let the function

$$f(z)=z-a_2z^2-a_3z^3 \tag{2.12}$$

be in the class $T^{*}(\mu)$, Then $f(z)$ is starlike in $|z|<r_3(\rho,\mu)$, where

$$r_3(\rho,\mu)=\sqrt{\frac{(1-\rho)(3-\mu)}{(3-\rho)(1-\mu)}} \tag{2.13}$$

In Figure 1, graph radius of starlike in the above example by Wolfram Alpha.

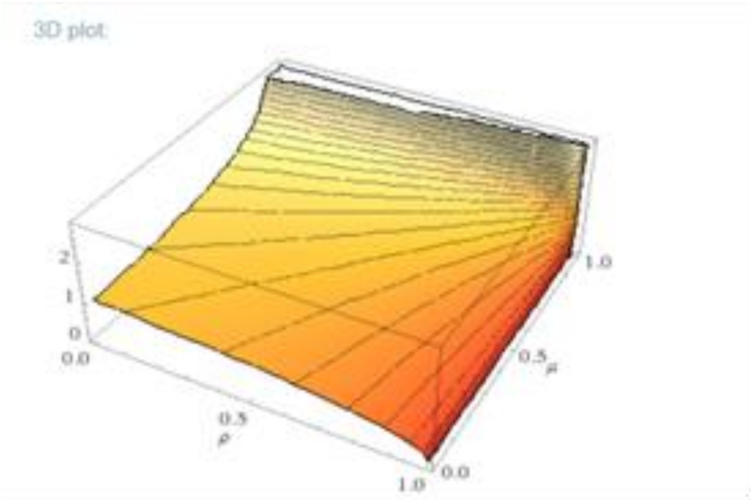

Figure-1, radius of starlike function defined by (2.12)

Put $\alpha=0$ in Theorem 2.1, we get the following corollary:

Corollary 2.2

Let the function $f(z)$ given by (1.1) be in the class $S_\lambda^*(\mu)$, then $f(z)$ is starlike in $|z|<r_4(k,n,\rho)$, where

$$r_4(k,n,\rho)=\inf_{k\geq 2}\left[\frac{(1-\rho)(k-\mu)\delta(n,k)}{(1-\mu)(k-\rho)}\right]^{\frac{1}{k-1}},\quad (k\geq 2). \tag{2.14}$$

The result is sharp for the function

$$f(z)=z-\frac{1-\mu}{(k-\mu)\delta(n,k)}z^k,\qquad (k\geq 2; z\in U).$$

(2.15)

Put $k=3$ in Corollary 2.2, we get the following example:

Example 2.2: Let the function $f(z)$ defined by (2.12) be in the class $S_\lambda^*(\mu)$, then $f(z)$ is stalike in $|z|<r_5(n,\rho,\mu)$, where

$$r_5(n,\rho,\mu)=\sqrt{\frac{(1-\rho)(3-\mu)(n+1)(n+2)}{2(3-\rho)(1-\mu)}} \tag{2.16}$$

The result is sharp for the function

$$f(z)=z-\frac{2(1-\mu)}{(n+2)(n+1)(3-\mu)}z^3,\qquad (z\in U).$$

(2.17)

Theorem 2.2:

Let $f(z)\in S_{\lambda,\alpha}^n(\mu)$. Then $f(z)$ is convex of order $\rho(0\leq\rho<1)$ in $|z|<r_6(n.,k,\lambda,\alpha,\mu,\rho)$, where

$$r_6\left(n.,k,\lambda,\alpha,\mu\right)=\inf_{k\geq 2}\left[\frac{(1-\rho)(k-\mu)\left[\alpha[1+(k-1)\lambda]^n+(1-\alpha)\delta(n,k)\right]}{k(k-\rho)(1-\mu)}\right]^{\frac{1}{k-1}}. \quad (2.18)$$

The result is sharp for the function

$$f(z)=z-\frac{1-\mu}{k(k-\mu)\left[\alpha[1+(k-1)\lambda]^n+(1-\alpha)\delta(n,k)\right]}z^k,\ (z\in U,k\geq 2). \quad (2.19)$$

Proof:
By using the same technique which used in the proof of Theorem 2.1, we can show that

$$\left|\frac{zf''(z)}{f'(z)}\right|<1-\rho \quad \text{for} \quad |z|<r_6$$

which give the assertion of Theorem 2.2.
Put $n=0$ in Theorem 2.2, we get the following corollary

Corollary 2.3:

Let the function $f(z)$ given by (1.2) be in the class $T^*(\mu)$, then $f(z)$ is convex of order $\rho(0\leq\rho<1)$ in $|z|<r_7(k,\rho,\mu)$, where

$$r_7(k,\rho,\mu)=\inf_{k\geq 2}\left[\frac{(1-\rho)(k-\mu)}{k(k-\rho)(1-\mu)}\right]^{\frac{1}{k-1}},\quad (k\geq 2). \quad (2.20)$$

The result is sharp for the function

$$f(z)=z-\frac{1-\mu}{k(k-\mu)}z^k,\quad (k\geq 2, z\in U). \quad (2.21)$$

Put $k=2$ and $\mu=0$ in Corollary 2.3, we get

Example 2.3: Let the function defined by (2.12) be in the class T^*, then $f(z)$ is convex in $|z|<r_8(\rho)$, where

$$r_8(\rho)=\frac{2(1-\rho)}{2(2-\rho)} \quad (2.22)$$

The result is sharp for the function

$$f(z)=z-\frac{1}{4}z^2,\quad (z\in U). \quad (2.23)$$

In figure 2 , graph the sharp function in Example 2.3 by Complex Tool program

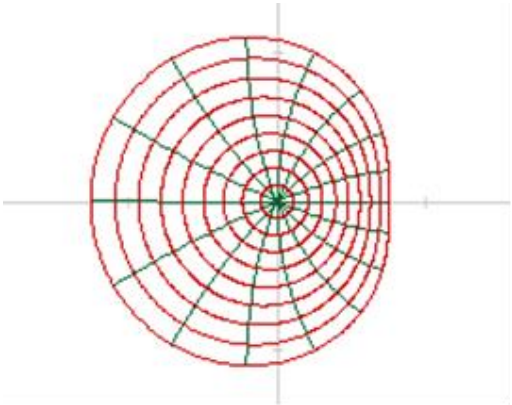

Figure 2: the image of unit disc under the function (2.23)

Put $k=3$ in Corollary 2.3, we get

Example 2.4: Let the function defined by (3.12) be in the class $T^*(\mu)$, then $f(z)$ is convex in $|z|<r_9(\rho,\mu)$, where

$$r_9(\rho,\mu)=\sqrt{\frac{(1-\rho)(3-\mu)}{3(3-\rho)(1-\mu)}} \tag{2.24}$$

The result is sharp for the function

$$f(z)=z-\frac{1-\mu}{3(3-\mu)}z^3, \quad (z\in U). \tag{2.25}$$

In Figure 3, graph the radius of convex in the above Example by Wolfram Alfa program, we get

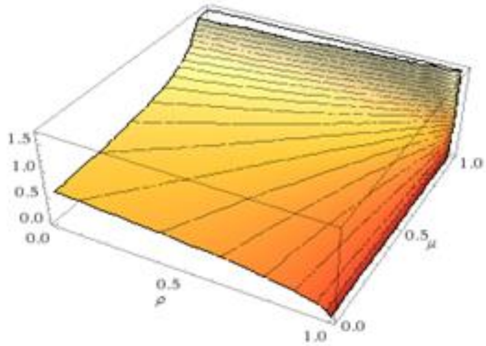

Figure 3: radius of convex function defined by (2.25)

Put $\rho=0$ in Corollary 2.3, we get the following corollary

Corollary 2.4

Let the function $f(z)$ given by (1.1) be in the class $T^*(\mu)$, Then $f(z)$ is convex in $|z|<r_9(k,\mu)$,

$$r_9\,(k,\mu)=\inf_{k\geq 2}\left[\frac{(k-\mu)}{k^2(1-\mu)}\right]^{\frac{1}{k-1}}. \tag{2.26}$$

The result is sharp for the function $f(z)$ given by (2.21).

RESULT

The result in Corollary 2.4 given the known result of Silverman [7, Theorem 8]

CONCLUSION

This work is a generalization for well-known radius problem of univalent functions and gave some examples.

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