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BINARY REPRODUCING KERNEL HILBERT SPACE APPROACH FOR SOLVING WICK TYPE STOCHASTIC KDV EQUATION

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ABSTRACT

In this work, we aim at applying an appropriate kernelization approach to solve analytically the Wick-type stochastic Korteweg-de Vries (KdV) equation with variable coefficients. Per the benefit of formulating the problem in Hilbert space, we deliver a binary reproducing kernel Hilbert space (RKHS) structure to represent the solution of such problem in the suggested kernel Hilbert space. Implying Hermite transform, white noise theory and proper binary reproducing kernel Hilbert spaces, we articulate white noise functional solutions for the Wick-type stochastic KdV equations. Representation of the exact solution is given in some reproducing kernel space. The uniform convergence, of the approximate solution together with its first derivative utilizing the suggested scheme, is investigated. The relevance of our suggested approach is inspected partially on one of the most important spectral density study, namely the cross power spectral density (CPSD) attests to the reliability of the scheme and highlighted the worth of the present work that can be applied on a wide class of nonlinear partial differential emerge in numerous physical modeling phenomena.

Keywords: Reproducing kernel; Inner product; Hermite transform; White noise theory; Wick-type; stochastic KdV equation; Spectral density.

INTRODUCTION

Lately, a few investigations have been done with an end goal to track down the analytical and numerical solutions of KdV differential equation:

$$D_t U + P(t) \diamond U \diamond D_x U + Q(t) \diamond D_x^3 U = F(x,t) \diamond \dot{W}(t) \quad , \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+ , \tag{1.1}$$

The term $F(x,t) \diamond \dot{W}(t)$ describes a state dependent random noise, where $w(t)_{t\in[0,t]}$ is an $\mathcal{F}_t$ adapted Wiener process defined in completed probability space (Ω, F, P) with expectation ε and associated with the normal filtration $\mathcal{F}_t = \sigma\{W(s); \quad 0 < s < t\}$, the coefficients P(t)and Q(t) are Gaussian white noise functions and "$\diamond$ " is the Wick product on the Kondratiev distribution space $(S)_{-1}$ defined in [1]. Eq. (1.1) can be considered as the Wick version of the following variable coefficients KdV equation:

$$\frac{\partial u}{\partial t}+p(t)u\frac{\partial u}{\partial x}+q(t)\frac{\partial^3 u}{\partial x^3}=f(x,t) \quad , \quad (x,t)\in\mathbb{R}\times\mathbb{R}_+ , \tag{1.2}$$

where $u(x,t)$ represents the velocity field of a fluid, the state $u(.)$ takes values in a separable real Hilbert space with some inner product, $p(t)$ and $q(t)$ are bounded measurable integrable functions on $\mathbb{R}_+$, this means that Eq. (1.1) can be regarded as the perturbation of Eq. (1.2). El Wakil et al [2] highlighted that, Eq. (1.2) is the mathematical model for small but finite amplitude electron-acoustic solitary waves in plasma of cold electron fluid with two different temperature isothermal ions. Thus, if this model is perturbed by Gaussian white noise, then Eq. (1.1) can be interpreted as the mathematical model for the resultant phenomenon. Firstly, the Korteweg- De Varies devised KdV equation in 1895, to model the Russell effect phenomena of Solitons such as shallow water surfaces. In the Nineteenth century, this model plays a significant role in many scientific applications such as solid-state physics, chemical kinetics, and dispersive wave phenomena in numerous fields of science; e.g. plasma physics, fluid dynamics, optics, quantum mechanics and spectral analysis etc. Many authors investigate this issue, e.g., Wadati [3] who first presented and examined stochastic KdV equation, and revealed the diffusion of solitons of the KdV equation under Gaussian noise. Recently, many authors turned to the study random waves which become an important subject of stochastic partial differential equations (SPDEs), e.g.,Xie [4, 5], Chen [6, 7], Ghany [8], Ghany et al [9–11] and others, have examined extensively the SPDEs. The present work is mainly devoted to explore the kernelization approach to find the reproducing kernel function of the differential equation (1.1) where, the Eigen functions u(x) satisfies the initial conditionu$(0) = 0$. Reproducing kernels were utilized interestingly at the beginning of the twentieth century [12, 13]. Geng [14] have suggested a new reproducing kernel Hilbert space method for solving nonlinear boundary value problem of fourth order. Zhang et al. [15] build reproducing kernel functions of polynomials form. Gumah et al. [16] have explored the solutions of uncertain Volterra integral equations by reproducing kernel Hilbert space method. Hashemi et al. [17] have resolved the Lane-Emden equation by formfitting a reproducing kernel method and group preserving scheme. Arqub et al. [18] have established the numerical solutions of Lane-Emden fractional differential equation by a precise investigated method. Reproducing kernel hypothesis has critical executions in differential and integral equations, probability and statistics [19, 20] and many other significant implementations. Recently, many researchers applied this theory for solving numerous mathematical models, e.g. Li and Cui [21] use reproducing kernel theory for producing a precise solution for a special class of nonlinear operator equations. In [22], Jiang and Lin employed the reproducing kernel theory to obtain an approximate solution of time-fractional telegraph equation. Geng formed a new RKHS to get a convergent series solution of fourth-order two-point boundary value problems in [23]. Arqub et al. exploited reproducing kernel theory for finding an approximate solution of Fredholm integro-differential equations in [24]. Bushnaq et al. [25] introduced a reproducing kernel method for fractional Fredholm integro-differential equations that results in a uniformly convergent approximate solution. Lately, periodic boundary value problem of two-point second-order mixed integro-differential equation is solved in [26–29]. Analytical solutions of some specific type of stochastic KdV differential equations with constant coefficients are acquired by several approaches in [30-31].

Here, our goal is finding an appropriate kernelization scheme to solve analytically the Wick-type stochastic KdV differential equations with variable coefficients and it is orderly as follows: Section 2 is faithful to introduce the essential used properties of the Hilbert space theory and suggested related RKHS. In section 3, we use Hermite transform to produce the suggested binary reproducing kernel Hilbert space for the considered KdV problem. Section 4 introduces the recommended representation of the KdV operator in equation (1.2). Section 5 is devoted to investigate the main results of this work, in which the white noise functional solution for (1.1) is formulated and ends with a declarative example tracked by a 3D figured soliton solution.Section 6, gives some thought on the applications of RKHS in wave optics, namely the cross power spectral density phenomena, to investigate the compatibility of the presented approach. Section 7, conclusions are vested.

PRELIMINARIES

Suppose that $S(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$are the Hida test function space and the Hida distribution space on$\mathbb{R}^d$, respectively. Let $h_n(x)$ be a Hermit polynomial, setting

$$\zeta_n = e^{-x^2} \; h_n(\sqrt{2x}) / \sqrt{((n-1)!\pi)}, n \geq 1. \tag{2.1}$$

Let $\alpha = (\alpha_1, \cdots, \alpha_d)$ denote d-dimensional multi-indices with$\alpha_1, \cdots, \alpha_d \in \aleph$. The family of tensor products

$$\zeta_\alpha \coloneqq \zeta(\alpha_1, \alpha_2, \ldots\ldots, \alpha_d) = \zeta_{\alpha_1} \otimes \zeta_{\alpha_2} \otimes \ldots\ldots \otimes \zeta_{\alpha_d}, \tag{2.2}$$

Forms an orthogonal basis for $L_2(\mathbb{R})$. Suppose that $\alpha^{(i)} = (\alpha_1^{(i)}, \alpha_2^{(i)}, \ldots\ldots, \alpha_d^{(i)})$ is the i[th] multi-index in some fixed ordering of all d-dimensional multi-indices α. We can, and will, assume that this ordering has the property of

$$i < j \Rightarrow \alpha_1^{(i)} + \alpha_2^{(i)} + \ldots\ldots + \alpha_d^{(i)} < \alpha_1^{j} + \alpha_2^{(j)} + \ldots\ldots + \alpha_d^{(j)}$$

i.e., the $\{\alpha^{(j)}\}_{j=1}^{\infty}$occurs in an increasing order. Now define

$$\eta_i \coloneqq \zeta_{\alpha_1^{(i)}} \otimes \zeta_{\alpha_2^{(i)}} \otimes \ldots\ldots \otimes \zeta_{\alpha_d^{(i)}}, i \geq 1.$$
(2.3)

We need to consider multi-indices of arbitrary length. For simplification of notation, we regard multi-indices as elements of the space $(N^N)_c$ of all sequences $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ with elements $\alpha_i \in N_0$ and with compact support, i.e., with only finitely many $\alpha_i \neq 0$. We write $J = (N^N)_c$, for $\alpha \in J$. Define

$$\eta_i \coloneqq \zeta_{\alpha_1^{(i)}} \otimes \zeta_{\alpha_2^{(i)}} \otimes \ldots\ldots \otimes \zeta_{\alpha_d^{(i)}}, i \geq 1.$$
(2.4)

We need to consider the multi-indices of arbitrary length. For simplification of notation , we regard multi-indices as elements of the space $(N_0^N)_c$ of all sequences $\alpha = (\alpha_1, \alpha_2, \ldots\ldots, \alpha_d)$ with elements $\alpha_i \in N_0$ and with compact support; that is, with only finitely many $\alpha_i \neq 0$. We write $J = (N_0^N)_c$, for $\alpha \in J$, we define

$$H_\alpha(\omega) \coloneqq \prod_{i=1}^{\infty} h_{\alpha_i}(<\omega, \eta_i>), \qquad \omega = (\omega_1, \omega_2, \ldots\ldots, \omega_d) \in S'(\mathbb{R}^d) \tag{2.5}$$

For a fixed $n \in N$ and for all $k \in N$, suppose that the space $(S)_1^n$ consists of those $f(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega) \in \oplus_{k=1}^n L_2(\mu)$, $c_\alpha \in \mathbb{R}^n$ such that

$$\|f\|_{1,k}^2 = \sum_\alpha c_\alpha^2 (\alpha!)^2 (2N)^{k\alpha} < \infty, \tag{2.6}$$

Where $c_\alpha^2 = |c_\alpha|^2 = \sum_{k=1}^{n} (c_\alpha^{(k)})^2$ *if* $c_\alpha = (c_\alpha^{(1)}, c_\alpha^{(2)}, \ldots\ldots, c_\alpha^{(n)}) \in \mathbb{R}^d, \mu$ is white noise measure on $(S'(\Re), B(S'(\Re))),\ \alpha! = \prod_{k=1}^{\infty} \alpha_k!$ *and* $(2\mathrm{N})^\alpha = \prod_j (2j)^{\alpha_j}$ *for* $\alpha \in J$. The space $(S)_{-1}^n$ consists of all formal expansions $F(\omega) = \sum_\alpha b_\alpha H_\alpha(\omega)$ *with* $b_\alpha \in \mathbb{R}^d$ such that $\|f\|_{-1,-q} = \sum_\alpha b_\alpha^2 (2\mathrm{N})^{-q\alpha} < \infty$ for some $q \in \mathrm{N}$. The family of seminorms $\|f\|_{1,k}, k \in \mathrm{N}$ gives rise to a topology on $(S)_1^n$, and we can regard $(S)_{-1}^n$ as the dual of $(S)_1^n$ by the action

$$\langle F, f \rangle = \sum_\alpha (b_\alpha, c_\alpha)\, \alpha! \tag{2.7}$$

Where (b_α, c_α) is the inner product in $\mathbb{R}^n$. The Wick product $f \lozenge F$ of two element $f = \sum_\alpha a_\alpha H_\alpha$, *and* $F = \sum_\beta b_\beta H_\beta \in (S)_{-1}^n$ *with* $a_\alpha, b_\beta \in \mathbb{R}^n$, is defined by

$$f \lozenge F = \sum_{\alpha,\beta} (a_\alpha, b_\beta)\, H_{\alpha+\beta}\ .$$
(2.8)

The space $(S)_1^n, (S)_{-1}^n$, $S(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$ are closed under Wick products. For $F = \sum_\alpha b_\alpha H_\alpha \in (S)_{-1}^n$, *with* $b_\alpha \in \mathbb{R}^n$, the Hermite transformation of F, is defined by

$$HF(z) = \tilde{F}(z) = \sum_\alpha b_\alpha z^\alpha \in \mathbb{C}^N, \tag{2.9}$$

Where $z = (z_1, z_2, \ldots\ldots) \in \mathbb{C}^N$ (the set of all sequences of complex numbers) and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \ldots\ldots z_n^{\alpha_n}$, *if* $\alpha \in J$, *where* $z_j^0 = 1$. For $F, G \in (S)_{-1}^n$ we have

$$F \tilde{\lozenge} G(z) = \tilde{F}(z).\tilde{G}(z), \tag{2.10}$$

For all z such that $\tilde{F}(z)$ and $\tilde{G}(z)$ exist. The produect on the right-hand side of the above formula is the complex bilinear product between two elements of $\mathbb{C}^N$ defined by $(z_1^1, z_2^1, \ldots\ldots, z_n^1).(z_1^2, z_2^2, \ldots\ldots, z_n^2) = \sum_{k=1}^n z_k^1 z_k^2$. Let $X = \sum_\alpha a_\alpha H_\alpha$, then the vector $c_0 = X(o) \in \mathbb{R}^n$ is called the generalized expectation of X , which is denoted by $E(X)$. Suppose that $g : U \to \mathbb{C}^M$ is an analytic function, where U is a neighborhood of $E(X)$. Assume that the Taylor series of g around $E(X)$ have coefficients in $\mathbb{R}^M$ .Then, the Wick version $g^\lozenge(X) = H^{-1}(g \circ \tilde{X}) \in (S)_{-1}^M$. In other words, if g has the power series expansion $g(z) = \sum a_\alpha (z - E(X))^\alpha$, with $a_\alpha \in \mathbb{R}^M$ then $g^\lozenge(z) = \sum a_\alpha (z - E(X))^{\lozenge\alpha} \in (S)_{-1}^M$.

The remainder of this section comprises some initial ideas on the Hilbert space hypothesis and reproducing kernels. We familiarize the following essential definitions and theorems from functional analysis related to Hilbert spaces and the RKHS, adapted to our main inquiry.

Definition2.1. Let X be a nonempty set. A function k: $X \times X \to C$ is called re- producing kernel function of the Hilbert space H if and only if

a) $k(\cdot, t) \in$ H for all $t \in X$,

b) $< \varphi, k(\cdot, t) >= \varphi(t)$ for all $t \in X$ and $\varphi \in$ H.

Any Hilbert space H is called reproducing kernel Hilbert space if there exists reproduc- ing kernel function k: $X \times X \to C$ for some nonempty set X. Let $AC[a, b]$ denotes the space of absolutely continuous functions on [a,b].

Theorem2.2. a) The space

$$\Gamma_2^1[a,b] := \{u \in AC[a,b]; \quad u' \in L_2[a,b]\} \tag{2.11}$$

is a reproducing kernel space and its reproducing kernel function is given by

$$\varphi_t(\eta) = \begin{cases} 1+\eta, & a \le \eta \le t \le b \\ 1+t, & a \le t \le \eta \le b \end{cases} \tag{2.12}$$

Where, the inner product and the norm in $\Gamma_2^1[a,b]$ are given by

$$\prec u, v \succ_{\Gamma_2^1} = u(a)v(a) + \int_a^b u'(\eta)v'(\eta)\, d\eta$$

b) The space

$$\Gamma_2^{1,0}[a,b] := \{u \in AC[a,b]; \quad u' \in L_2[a,b], u(0) = 0\}$$

is a reproducing kernel space and its reproducing kernel function is given by

$$\Psi_t(\xi) = \begin{cases} \xi, & a \le \xi \le t \le b \\ t, & a \le t \le \xi \le b \end{cases}$$

Where, the inner product and the norm are given by

$$\prec u, v \succ_{\Gamma_2^{1,0}} = u(a)v(a) + \int_a^b u'(\xi)v'(\xi)\, d\xi$$

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c) The space

$$\Gamma_2^{4,0}[a,b] = \begin{cases} u \in AC[a,b], & u, u', u'', u''' \in AC[a,b] \\ u^{(4)} \in L_2[a,b], & u(a) = u'(a) = u(b) = u'(b) = 0 \end{cases} \tag{2.13}$$

is a reproducing kernel space and its reproducing kernel function is given by

$$\Phi_t(\xi) = \begin{cases} \sum_{i=1}^{8} c_i(t)\xi^{(i-1)}(t), & a \le \xi \le t \le b \\ \sum_{i=1}^{8} d_i(t)\xi^{(i-1)}(t), & a \le t \le \xi \le b \end{cases} \tag{2.14}$$

Where, the inner product and the norm in $\Gamma_2^{4,0}[0,1]$ are given by

$$\prec u,v \succ_{\Gamma_2^{4,0}} = \sum_{i=0}^{3} u^{(i)}(a)v^{(i)}(a) + \int_a^b u^{(4)}(\eta)v^{(4)}(\eta)\, d\eta \tag{2.15}$$

Proof. The proof of the first two items can be found in [32], from definition of the inner- product of the space $\Gamma_2^{4,0}[0,1]$ given by equation (2.15) we have

$$\begin{aligned}\prec v,\phi_t \succ_{\Gamma_2^{4,0}[0,1]} &= u(0)\phi_t(0) + u'(0)\phi'_t(0) + u''(0)\phi''_t(0) + u'''(0)\phi''_t(0)\\ &+ u^{(3)}(1)\phi_t^{(4)}(1) - u^{(2)}(1)\phi_t^{(5)}(1) + u^{(1)}(1)\phi_t^{(6)}(1) - u(1)\phi_t^{(7)}(1)\\ &- u^{(3)}(0)\phi_t^{(4)}(0) + u^{(2)}(0)\phi_t^{(5)}(0) - u^{(1)}(1)\phi_t^{(6)}(1) + u(1)\phi_t^{(7)}(1)\\ &+ \int_a^b u(\eta)\phi_t^{(8)}(\eta)\, d\eta\end{aligned}$$

Since, $\Gamma_2^{4,0}[0,1]$, so

$$\phi_t(0) = \phi'_t(0) = \phi_t(1) = \phi'_t(1) = 0 \tag{2.16}$$

If we have,

$$\Phi_t(\xi) = \begin{cases} \phi_t^{(3)}(0) - \phi_t^{(4)}(0) = 0 \\ \phi_t^{(2)}(0) + \phi_t^{(5)}(0) = 0 \\ \phi_t^{(5)}(1) = \phi_t^{(4)}(1) = 0 \end{cases} \tag{2.17}$$

This implies,

$$\prec v,\phi_t \succ_{\Gamma_2^{4,0}[0,1]} = \int_a^b u(\eta)\phi_t^{(8)}(\eta)\, d\eta$$

Hence,

$$\phi_t^{(8)}(\eta) = \delta(\eta - t) \tag{2.18}$$

And

$$\Phi_t(\xi) = \begin{cases} \sum_{i=1}^{8} c_i(t)\xi^{(i-1)}(t), & a \le \xi \le t \le b \\ \sum_{i=1}^{8} d_i(t)\xi^{(i-1)}(t), & a \le t \le \xi \le b \end{cases} \tag{2.19}$$

Using equations (2.16)-(2.18) in equation (2.19), we can easily calculate the exact values of the constants $c_i(t)$ and $d_i(t)$, i=1,2,...,8.

BINARY REPRODUCING KERNEL

Taking Hermite transform of (1.1) we get the following:

$$\widetilde{U_t} + \tilde{P}(t)\tilde{U}\widetilde{U_x} + \tilde{Q}(t)\widetilde{U_{xxx}} = 0 \;\; , \;\; (x,t) \in \mathbb{R} \times \mathbb{R}_+ \;, \tag{3.1}$$

For simplicity, we denote $\widetilde{U}$ by u, so equation (3.1) can be written in the form (1.2). To solve equation (1.2), we need to define a binary reproducing kernel Hilbert space. Let $CC(D)$ denote
the space of completely continuous functions on D = [0, 1] × [0, 1] and define the binary reproducing kernel Hilbert space ${}^{0}\Gamma_2^{(2,4)}$(D)[32] by:

$$ {}^{0}\Gamma_2^{(2,4)}(D) = \begin{cases} v: \quad \dfrac{\partial^4 v}{\partial^3 x \partial t} \in CC(D), \quad \dfrac{\partial^6 v}{\partial^4 x \partial^2 t} \in L_2(D) \\ v(x,0) = v(0,t) = v(1,t) = v'(0,t) = v'(1,t) = 0 \end{cases} $$

The inner product and the norm of this space are given by [32]:

$$ \prec u, v \succ_{{}^{0}\Gamma_2^{(2,4)}(D)} = \sum_{i=0}^{3} \int_0^1 \frac{\partial^{i+2} u(0,,t)}{\partial^i x \partial^2 t} \times \frac{\partial^{i+2} v(0,,t)}{\partial^i x \partial^2 t} \, dt + \sum_{j=0}^{1} \prec \frac{\partial^j u(.,,t)}{\partial^j t}, \frac{\partial^j v(.,,t)}{\partial^j t} \succ_{\Gamma_2^2 [0,1]} + \int_0^1 \int_0^1 \frac{\partial^6 u(x,,t)}{\partial^4 x \partial^2 t} \times \frac{\partial^6 v(x,,t)}{\partial^4 x \partial^2 t} \, dx dt $$

and

$$ \| u \|_{{}^{0}\Gamma_2^{(2,4)}(D)} = \sqrt{\prec u, u \succ_{{}^{0}\Gamma_2^{(2,4)}(D)}}, \quad u, v \in {}^{0}\Gamma_2^{(2,4)}(D). $$

Theorem3.1. The reproducing kernel function of the space ${}^{0}\Gamma_2^{(2,4)}(D)$ is given by [32]:
$\Omega(x,t) = \Phi_t(\xi)\Psi_x(\xi)$

Similar to the above definition, we define the binary reproducing kernel Hilbert space

${}^{0}\Xi_2^{(1,1)}(D)$ [32] by:

$$ {}^{0}\Xi_2^{(1,1)}(D) := \{ v \in CC(D) : \quad \frac{\partial v}{\partial x} \in CC(D), \frac{\partial^2 v}{\partial x \partial t} \in L_2(D) \} $$

The inner product and the norm of this space is given by[32]:

$$\prec u,v\succ_{{}^{0}\Xi_2^{(1,1)}(D)}=\int_0^1 \frac{\partial u(0,,t)}{\partial t}\times\frac{\partial v(0,,t)}{\partial t}\,dt+\int_0^1\int_0^1 \frac{\partial^2 u(x,,t)}{\partial x\partial t}\times\frac{\partial^2 v(x,,t)}{\partial x\partial t}\,dxdt+\prec u(x,,1),v(x,,1)\succ_{\Gamma_2^1}$$

and

$$\| u\|_{{}^{0}\Xi_2^{(1,1)}(D)}=\sqrt{\prec u,u\succ_{{}^{0}\Xi_2^{(1,1)}(D)}},\quad u,v\in {}^{0}\Xi_2^{(1,1)}(D).$$

Theorem3.2. The reproducing kernel function of the space ${}^{0}\Xi_2^{(1,1)}(D)$ is given by[32]:
$\Omega(x,t)=\phi_t(\xi)\psi_x(\xi)$

For simplicity, we will denote ${}^{0}\Gamma_2^{(2,4)}(D)$ by Γ (D) and ${}^{0}\Xi_2^{(1,1)}(D)$ by Ξ (D).

REPRESENTATION OF THE KDV OPERATOR

In this section, we will give the matrix representation of the KdV equation (1.2) in the Hilbert space Γ(D). Let Δ = [0, X] × [0, T] and consider the bounded linear operator:

$$\mathcal{A}:\Gamma(\Delta)\to\Xi(\Delta)$$

Given by

$$\mathcal{A}u=\frac{\partial u}{\partial t}+q(t)\frac{\partial^3 u}{\partial x^3} \tag{4.1}$$

Then the KdV equation (1.2) can be rewritten in the form:

$$\begin{cases}\mathcal{A}u=f(x,t,u)-p(t)u\frac{\partial u}{\partial x}, & z:(x,t)\in\Delta,\\ u(0)=0.\end{cases} \tag{4.2}$$

Put $\rho_i(z)=\mathcal{A}_{z_i}(z)$ and $\sigma_i(z)=\mathcal{A}^*\rho_i(z)$, where $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$. Using the Gram-Schmidet orthogonalization process of $\{\sigma_i(z)\}_{i=1}^{\infty}$, we will obtain the orthonormal system $\{\tilde{\sigma}_i(z)\}_{i=1}^{\infty}$ [33]:

$$\{\tilde{\sigma}_i(z)\}_{i=1}^{\infty}=\sum_{j=1}^{i}a_{ij}\,\sigma_i(z),\quad a_{ij}>0, i=1,2,.... \tag{4.3}$$

REPRESENTATION AND SOLUTION PROPERTIES OF EQN.(1.2)

Lemma 5.1. For every dense sequence $\{z_i\}_{i=1}^{\infty}$ on Δ, the system $\{\sigma_i(z)\}_{i=1}^{\infty}$ is a complete system of Γ(Δ) and $\sigma_i(z)=\mathcal{A}\Omega_z(\xi)|_{\xi=z_i}$.

Proof. We have

$$\sigma_i(z) = \mathcal{A}^*\rho_i(z) = < \mathcal{A}^*\rho_i(\xi), \Omega_z(\xi) >$$

$$= < \rho_i(\xi), \mathcal{A}_\xi \, \Omega_z(\xi) > = A\Omega_z(\xi)|_{\xi=z_i}$$

We use $\mathcal{A}_\xi$ to indicate the operator $\mathcal{A}$ applies to the function of ξ. Let $u(z) \in \Gamma(\Delta)$ and

$$< u(z), \sigma_i(z) > = 0, i = 1,2, \dots$$

For all $\sigma_i(z) \in \Gamma(\Delta)$, this implies

$$< u(z), \mathcal{A}^*\rho_i(z) > = < A u(z), \rho_i(z) > = (\mathcal{A}u)(z_i) = 0$$

Since $\{z_i\}_{i=1}^{\infty}$ is dense on Δ, so $(\mathcal{A}u)(z_i) = 0$ implies $u = 0$.

Corollary 5.2. The sequence $\{ \tilde{\sigma}_i(z)\}_{i=1}^{\infty}$ is the complete basis of $\Gamma(\Delta)$.

Theorem 5.3. For every dense sequence $\{z_i\}_{i=1}^{\infty}$ on Δ, the solution of the KdV equation (1.2) is given by:

$$y(x,t) = \sum_{i=1}^{\infty} \sum_{j=1}^{i} a_{ij} \, f(x_j, t_j, u(x_j, t_j)) \tilde{\sigma}_i(x,t), \quad a_{ij} > 0, i = 1,2, \dots \tag{5.1}$$

Proof. Applying the reproducing property

$$< \tau(z), \rho_i(z) > = \tau(z_i),$$

For all $\tau \in \Gamma(\Delta)$. This implies

$$y(z) = \sum_{i=1}^{\infty} < y(z), \tilde{\sigma}_i(z) >_{\Gamma(\Delta)} \tilde{\sigma}_i(z)$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} a_{ik} < y(z), \mathcal{A}^*\Omega_{z_k}(z) >_{\Gamma(\Delta)} \tilde{\sigma}_i(z)$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} a_{ik} < A\, y(z), \Omega_{z_k}(z) >_{\Gamma(\Delta)} \tilde{\sigma}_i(z)$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} a_{ik} < f(z, y(z)), \Omega_{z_k}(z) >_{\Gamma(\Delta)} \tilde{\sigma}_i(z)$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} a_{ik} \, f(z_k, y(z_k)) \, \tilde{\sigma}_i(z)$$

Corollary 5.4. The approximate solution

$$y_N(x,t) = \sum_{i=1}^{N} \sum_{j=1}^{i} a_{ij} \, f(x_j, t_j, u(x_j, t_j)) \tilde{\sigma}_i(x,t), \quad a_{ij} > 0, i = 1,2, \dots \tag{5.2}$$

and its first derivative $y'_N(z)$ are uniformly convergent in Δ.

Proof. Firstly, we will prove that, for every solution y (z) in the space $\Gamma(\Delta)$, there exists a constant E > 0 such that:

$$||y^i(z)||_{C_\Delta} \leq E||y^i(z)||_{\Gamma(\Delta)}, \; i = 1,2,...$$

Where, $||y^i(z)||_{C_\Delta} = max_{z\in\Delta}|y(z)|$. Since

$$y^i(z) = |< y(t), \partial^i\Omega_z(t) >_{\Gamma(\Delta)}, \; i = 0,1$$

For all z ∈ Δ. So,

$$|y^i(z)| = ||y(t)||_{\Gamma(\Delta)}||\,\partial^i\Omega_z(t)||_{\Gamma(\Delta)}, \; i = 0,1$$

This implies

$$|y^i(z)| \leq E_i||y(t)||_{\Gamma(\Delta)}, \; i = 0,1$$

Where E_0 and E_1 are positive constants. Therefore

$$||y^i(z)||_{C_\Delta} \leq \max\{E_0, E_1\}||y^i(t)||_{\Gamma(\Delta)}, \; i = 1,2,...$$

We have

$$|y_N^{(i)}(z) - y^{(i)}(z)| = |< y_N(z) - y(z), \partial^i{}_z\Omega_z(t) > |_{\Gamma(\Delta)}$$

$$\leq ||y_N(z) - y(z)||_{\Gamma(\Delta)}||\,\partial^i{}_z\Omega_z(t)||_{\Gamma(\Delta)}$$

$$\leq E_i||y_N(z) - y(z)| \quad |_{\Gamma(\Delta)}, \; i = 0,1.$$

Obviously, Equation (5.1) implies that the solution of equation (1.1) can be written in the form

$$U(x,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{i} B_{ij}\, f^{\diamond}(x_j, t_j, u^*(x_j, t_j))\widetilde{\sigma_1}(x,t), \quad a_{ij} > 0, i = 1,2, ... \tag{5.3}$$

Without loss of generalization for f(x, t) = 0, the equation (1.2) has infinite number of Solutions, for example we consider the following two solutions:

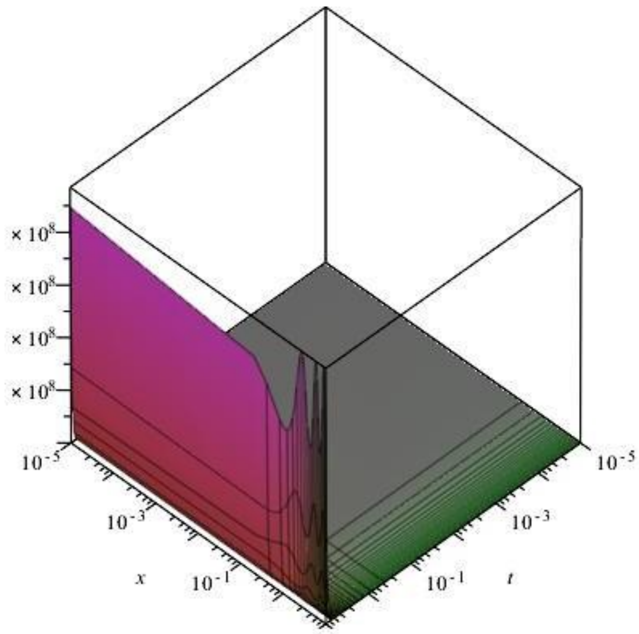
$$u_1 = \frac{\exp(c_1 t)}{\cosh[\cos(x - c_2 t)]} \tag{5.4}$$

$$u_2 = 0.5c_3\text{sech}[0.5\sqrt{c_3}\,(x - c_3 t)] \tag{5.5}$$

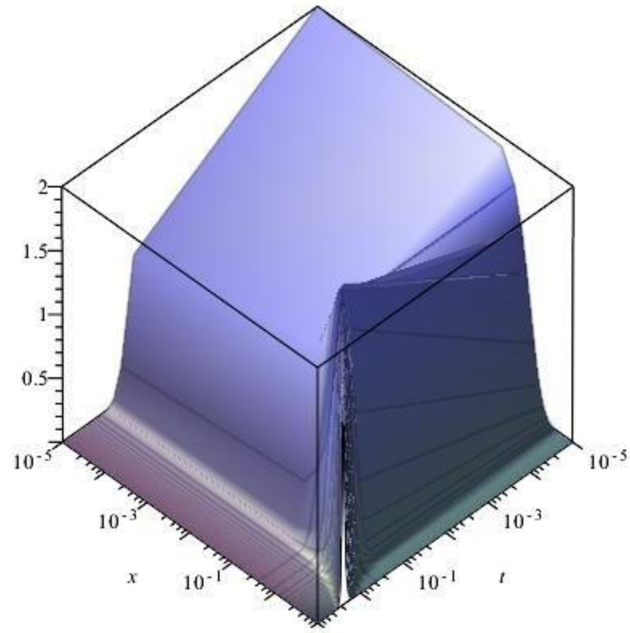
Where, c_1, c_2 and c_3 are real or complex constants.Using inverse Hermite transforms to (5.4) and (5.5) we obtain the following two solutions for (1.1):

$$U_1 = \frac{\exp^{\diamond}(c_1 t)}{\cosh^{\diamond}[\cos^{\diamond}(x - c_2 t)]} \tag{5.6}$$

$$U_2 = 0.5c_3\text{sech}^{\diamond}[0.5 \diamond \sqrt{c_3}\,(x - c_3 t)] \tag{5.7}$$



[a]



[b]

Figure 1: (a) 3D plot of the solution (5.5) and (b) 3D plot of the solution (5.6)

For the stochastic equation (1.1), we have acquired some of stable and consistent solu- tions which are represented in (5.7)-(5.8) as a stochastic version of (5.5)-(5.6). Since each result involves some open parameters, so we might achieve different type of general soliton shapes i.e., a self - reinforcing wave packet that maintain its shape while it propagates at a constant velocity. We have successfully figured our soliton solution associated with free parameters (Fig (a) and Fig (b)).

APPLİCATİON: RKHS İN SİGNAL PROCESSİNG

Here, we explore the importance of RKHS in optics, by discussing the cross power spectral density (CPSD) or cross-spectral density (CSD). Primarily, according to Moore-Aronszajn theorem [34, 35], every RKHS possesses anon-negative definite reproducing kernel $k(x,t)$ and contrariwise each non-negative definite kernel specifies an RKHS.As highlighted in [36], any cross power spectral density (CPSD) must be nonnegative definite, hence we can claim that any CPSD identifies a RKHS and vice versa. This implies that the awareness of the RKHS accompanying to a certain CPSD provides an access to the consistency features of the original optical fields. Similar to our technique in the previous sections of this paper, the integral equations associated to CPSDs can be solved explicitly, to give the appropriate eigen functions and eigen values. Conversely, an assortment of reproducing kernels carefully examined in math that did not infiltrate into optics. Szegö's and Bergman's [37] kernels,

$$k(x_1,x_2)=\frac{1}{1-x_1x_2^*} \quad \text{and} \quad k(x_1,x_2)=\frac{1}{(1-x_1x_2^*)^2}$$

are instances of this and could merit consideration from the optics specialists. Allow us to allude to a substantial model: in Eq. (2.14), we have a case in which the inner product of the two signals requires one to integrate the product of their first derivatives. It does not appear to be that incredible hardships ought to emerge in changing over such an activity in a test methodology, probably merging analogical and numerical strategies. Comparative prospects exist for quite a long time tasks engaged with RKHS examination. This bears the cost of a rich field of activity to the experimentalist notwithstanding the hypothetical subjects treated previously.

CONCLUSİON

In the present work, we scrutinize the solution for a wide class of the most important nonlinear partial differential equation arises in frequent physical mathematical modelling, mainly the Korteweg-de Vries (KdV) equation.The KdV type equation have third/fifth order dispersive term in their treatment that reflects propagation property near critical angle [38]. Therefore, KdV has a weighty role in the study of wave propagation [39]. It is truly challenging to track down the analytical solutions of such physical problems when these are vastly nonlinear. Generally, this nonlinearity cause that the exact solution may not be accessible and one should apply a suitable techniques to beat the nonlinear nature. Here we attack this problem by applying the proposed reproducing kernel Hilbert space (RKHS) scheme to deduce the exact and approximated solutions of third order stochastic KdV differential condition with variable coefficients problem. The scheme is based on utilizing diversity forms of the standard Hermite transform, the white noise functional theory, the matrix representation of the generalized KdV operator and the Gram-Schmidt orthogonalization process are used to generate an appropriate binary reproducing kernel Hilbert space (RKHS) for the under study problem. The obtained dense set of orthonormal sequences in an inner product Hilbert space construct the main core solution of the considered stochastic KdV problem.We establish the validity of the suggested binary RKHS generating the solution of the stochastic KdV differential equation with variable coefficients problem. The uniform convergence of the approximate solution and its first derivative retaining the proposed scheme are attested. The accommodated example in Section 5 followed by the given 3D figured soliton solution together with the perceived application (CPSD) in Section 6 reflect the effortlessness and featured the meaning of this work that can be enormously applied to other nonlinear problems in physics and engineering. The suggested scheme reproduces some fundamental distinction with the other current comparable strategy dealing with same KdV problem.

CONFLICT OF INTEREST

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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