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ALPHA LOGARITHM TRANSFORMED SEMI LOGISTIC DISTRIBUTION USING MAXIMUM LIKELIHOOD ESTIMATION METHOD

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ABSTRACT

In this article, we review the maximum likelihood method for estimating the parameters of a fitted model and show that this method generally provides the asymptotically best estimate with the smallest mean Error. Therefore, maximum likelihood estimation is sufficient for most applications in data science. The Fisher data matrix describes the orthogonality of parameters in a probabilistic model and always results from the highest possible estimate. Parameters associated with the model were estimated using the Maximum Likelihood Estimation (MLE) method. The maximum likelihood estimation method in a risk function is used to estimate the parameters of the alpha log-transformed semi-logistic distribution to determine the best method. Since the inverse of the Fisher data matrix provides the variance matrix of the prediction error, orthogonalizing the parameters ensures that the parameters are distributed independently of each other. Finally, the extended model was applied to real data and results showing the performance of ALTSL classification compared to other classification methods are presented. We present the MLE of the unknowns in this distribution using Newton-Raphson. We also calculate the Average Estimation (AE), Variance (VAR), Mean Absolute Deviation (MAD), Mean Square Error (MSE), Relative Absolute Bias (RAB) and Relative Efficiency (RE) for both the parameters under sample based on 10000 simulations to assess the performance of the estimators. Also, we derive the asymptotic confidence bounds for unknown parameters.

Keywords : ALTSL, MLE, Average Estimate (AE), Variance (VAR), Mean Absolute Deviation (MAD), Mean Square Error (MSE), Relative Absolute Bias (RAB) and Relative Efficiency (RE), Asymptotic confidence bounds.

INTRODUCTION

In a research study, obtaining a new distribution by adding additional parameters or jointly expanding the distribution using generators Sanku Dey et al. (2017) the purpose of this change is to add more detail to the classical distribution to aid in the analysis of mixed data. Madukar et al. (1993) and Marshall et al. (1997) developed a method to add new parameters to existing distributions. Eugene (2002) proposed the concept of beta-generating distributions where the principal distribution is beta and the root distribution may be the common factor distribution of all continuous variables. Jones et al. (2009) changed the minds of Eugene et al. (2002) by replacing the beta distribution with the Kumaraswamy distribution. Additionally, Alzaatreh et al. (2013) proposed a T-X series of continuous distributions in which the probability density function (pdf) of the beta distribution is replaced by the pdf of each continuous variable and converted to cdf. , the function of cdf is satisfied using certain conditions. Lee et al. (2013) provides detailed information on methods for creating regular distributions.

Recently, Mahdavi et al. (2017) proposed a new method called Alpha Power Transformation (APT) for incorporating additional parameters in continuous transmission. In fact, the goal is to integrate skewness into the central distribution. This transformation has been used by different researchers to obtain the Alpha power transform, including the general exponential distribution of the Alpha power transform by Sanku Dey et al. (2017), Alpha power conversion Lindly distribution and Alpha power conversion continuum by Sanku Dey et al. (2019) Exponential distribution from Hassan et al. (2018) transformed the inverse Lindly distribution by Alpha Power, Sanku Dey et al. (2019). Actuarial studies often use quasi-logistic distributions and Pareto distributions to model compensation. Semi-logical distributions have many uses in testing the lifespan of products based on the age of the product.
Pareto distribution is a well-known distribution used to model heavy tailed phenomena by Lee et al (2018). It also many applications in actuarial science, survival analysis, economics, life testing, hydrology, finance, telecommunication, reliability analysis, physics and engineering studied by Brazauskas et al (2003), Farshchian et al (2010) and Korkmaz et al (2018). Pareto distribution is successfully used by Philbrick et al (1985) for ridge of losses in an insurance company, real state and accountability experience of hospitals. Levy et al (1997) used Pareto distribution for investigation of wealth in society. Castillo et al (1997) considered generalized form of Pareto distribution to model exceedances over a margin in flood control. Various Pareto distributions and their generalizations can be found in the literature. Insurance payment data is often skewed and presents a broad-tailed distribution. The disadvantage of using the Pareto model for actuarial data is that it covers the behavior of very large losses, but not small losses, which can be represented by partial distribution, distribution negative output, gamma distribution or Weibbian distribution. The forest section is well modelled. Some other evolutionary changes Sanku Dey et al. (2017) introduced alpha power exponential (APE) and alpha power transformed Weibull (APTW) distributions, respectively. Sunku Dey et al. (2017) introduced a new family of three alpha log transformation indices and their applications.

Logarithmic transformation is widely used to handle skewed data in biomedical and psychosocial research. Changyong FENG et al. (2014) focus on the logarithmic transformation and discuss the main difficulties in using this model in practice. Abd-Elfattah (2006) investigated the effectiveness of the maximum estimator under censored sampling variance for semi-logistic distributions. Cheng and Chen (1988) derived conditions for the lifetime and specificity of the maximum estimator in the 2-parameter Weibull distribution, where the group profile is based on the number of contributing group names and group restrictions. Lindley (1950) introduced some group corrections to estimate the maximum number of parameters for pooled samples. Lloyd (1952) obtained estimates of the position and parameter values of order statistics using the least squares estimation method. Ramamohan and Anjaneyulu (2011) investigated how the least squares method can be effective in estimating the parameters of the two-parameter Weibull distribution from optimally constructed cluster models. Ramamohan and Anjaneyulu (2013) studied the use of least squares to estimate the parameter (σ) from an optimized model when the shape (β) is unknown.

Ramamohan and Anjaneyulu (2014) studied the estimation of parameter values (σ) from the optimal design using the square of the smallest margin distance estimation method when the image parameter (β) in the logistic distribution is known. Vijaya lakshmi, Raja Sekharam, and Anjaneyulu (2018) studied the estimation of scale (λ) and position (μ) of two-dimensional Rayleigh distribution using the averaging method. Vijaya lakshmi, Raja Sekharam and Anjaneyulu (2019) investigated the estimation of the scale

(q) and shape (α) parameters of the distribution by the least squares method using the data set. Vijaya lakshmi and Anjaneyulu (2019) studied the estimation of location (μ) and scale (λ) of two-parameter semi-logistic Pareto distribution (HLPD) by least squares regression method. Vijaya lakshmi and Anjaneyulu (2019) studied the estimation of location (μ) and scale (λ) of two-parameter semi-logistic Pareto distribution (HLPD) through mean level regression. K. V. Subrahmanyam et al. (2020) presented two new Alpha log-transformed Rayleigh (ALTR) distributions. K. V. Subrahmanyam et al. (2020) presented two new Alpha log-transformed Rayleigh distributions and their properties.

In this article, we discuss the estimation process of the unknowns of the ALTSL distribution. There are many estimation techniques in the literature, but the most popular estimation technique is the maximum likelihood estimator (MLE). The idea behind maximum parameter estimation is to consider the parameters that are most likely to be useful given the sample data. . We immediately took two data sets and followed the pattern shown to us. We simulate the data using a Monte Carlo simulation program and propose a maximum estimate of the uncertainty of the ALTSL distribution. We also calculated the Average Mean (AE), Variance (VAR), Standard Deviation (SD), Mean Absolute Deviation (MAD), Mean Square Error (MSE), Relative Absolute Bias (RAB), and Relative Error (RE) between two parameters. Measure the performance of the interpreter based on 10,000 simulation samples. We also provide asymptotic confidence bounds for uncertainty. The resulting Probability density, distribution, Survival and Hazard functions are derived from ALTSL.

A random variable X ~ ALTSL $(m; \theta, \sigma^2, \alpha)$ has probability density function and is in the form

$$f_{ALTSL}(m; \theta, \sigma^2, \alpha) = \frac{2(\alpha-1)e^{m}}{[\sigma(1+e^{m})^2]\left[\log\left\{\alpha-(\alpha-1)\left[\frac{1-e^{-m}}{1+e^{-m}}\right]\right\}\right]}, if \alpha > 0, \alpha \neq 1, m > 0 \quad \ldots(1.1)$$

$(\theta, \sigma^2, \alpha)$ are location scale and shape parameters

A random variable X~ ALTSL $(m; \theta, \sigma^2, \alpha)$ has cumulative distribution function and is in the form

$$F_{ALTSL}(m; \theta, \sigma^2, \alpha) = 1-\frac{log\left\{\alpha-(\alpha-1)\left[\frac{1-e^{-m}}{1+e^{-m}}\right]\right\}}{\log\alpha}, if \alpha > 0, \alpha \neq 1, m > 0 \quad \ldots(1.2)$$

$(\theta, \sigma^2, \alpha)$ Are location scale and shape parameters and m = $\frac{(x-\theta)}{\sigma}$

A random variable X ~ ALTSL $(m; \theta, \sigma^2, \alpha)$ has Quantile function and is in the form
The p^{th} quantile x_p of ALTSL distribution is the root of the equation

$$x_p = \sigma\sqrt{2log\left[\frac{\alpha-\alpha^{(1-p)}}{(\alpha-1)}-1\right]} \quad \ldots(1.3)$$

RANDOM NUMBER GENERATION

Let U~ U(0,1), then equation (2.1.1) can be used to simulate a random sample of size n from the ALTSL distribution as follows

$$m_i = \sigma\sqrt{2log\left[\frac{\alpha-\alpha^{(1-u_i)}}{(\alpha-1)}-1\right]}, I = 1, 2, \ldots, n. \quad \ldots(1.4)$$

2.1 ESTIMATION OF PARAMETERS OF ALTSL DISTRIBUTION MAXIMUM LIKELIHOOD METHOD

Let $m_1, m_2, \ldots, m_n$ be a random sample of size 'n' from ALTSL $(m; \theta, \sigma^2, \alpha)$ then the likelihood function L of this sample is defined as

$$l_n = ln(f_{ALTSL}(m; \theta, \sigma^2, \alpha))$$

$$= n\ln 2 + n\ln(\alpha-1) + \sum_{i=1}^{n} m_i \, loge - nlog\sigma - 2\sum_{i=1}^{n} log(1+e^{m_i}) - - \sum_{i=1}^{n} ln(\alpha - (\alpha-1)([1-e^{m_i}]/1+e^{m_i})) \quad \ldots (2.1.1)$$

Calculating the 1st and 2nd order partial derivative of (2.1.1) with respective to (θ, σ, α) and then 1st order partial derivatives equating to zero we get the following equations

$$\frac{dln}{d\theta} = \frac{\frac{-(\alpha-1)e^{-\frac{x-\theta}{\sigma}}\left(e^{-\frac{x-\theta}{\sigma}}+1\right)}{\sigma\left(1-e^{-\frac{x-\theta}{\sigma}}\right)^2} - (\alpha-1)e^{-\frac{x-\theta}{\sigma}}}{\left(\alpha - \frac{(\alpha-1)\left(e^{-\frac{x-\theta}{\sigma}}+1\right)}{\left(1-e^{-\frac{x-\theta}{\sigma}}\right)}\right)\ln\left(\alpha - \frac{(\alpha-1)\left(e^{-\frac{x-\theta}{\sigma}}-1\right)}{\left(1-e^{-\frac{x-\theta}{\sigma}}\right)}\right)} = 0 \quad \ldots (2.1.2)$$

$$\frac{dln}{d\sigma} = \frac{2(\alpha-1)e^{\frac{x-\theta}{\sigma}}}{\sigma^2\left(e^{\frac{x-\theta}{\sigma}}-1\right)\left(e^{\frac{x-\theta}{\sigma}}-2\alpha+1\right)\ln\left(\alpha - \frac{(\alpha-1)\left(e^{-\frac{x-\theta}{\sigma}}+1\right)}{\left(1-e^{-\frac{x-\theta}{\sigma}}\right)}\right)} = 0 \quad \ldots (2.1.3)$$

$$\frac{dln}{d\alpha} = \frac{2}{\sigma^2\left(e^{\frac{x-\theta}{\sigma}}-1\right)\left(2\alpha-e^{\frac{x-\theta}{\sigma}}-1\right)\ln\left(\alpha - \frac{(\alpha-1)\left(e^{-\frac{x-\theta}{\sigma}}+1\right)}{\left(1-e^{-\frac{x-\theta}{\sigma}}\right)}\right)} = 0 \quad \ldots (2.1.4)$$

2nd order partial derivative is given by

$$\frac{d^2ln}{d\theta^2} = \frac{2(\alpha-1)e^{\frac{x-\theta}{\sigma}}}{\sigma\left(e^{\frac{x-\theta}{\sigma}}-1\right)\left(e^{\frac{x-\theta}{\sigma}}-2\alpha+1\right)\ln\left(\alpha - \frac{(\alpha-1)\left(e^{-\frac{x-\theta}{\sigma}}+1\right)}{\left(1-e^{-\frac{x-\theta}{\sigma}}\right)}\right)} \quad \ldots (2.1.5)$$

$$\frac{d^2ln}{d\sigma^2} = \frac{2(\alpha-1)(x-\theta)e^{\frac{x-\theta}{\sigma}}}{\sigma^2\left(e^{\frac{x-\theta}{\sigma}}-1\right)\left(e^{\frac{x-\theta}{\sigma}}-2\alpha+1\right)\ln\left(\alpha - \frac{(\alpha-1)\left(e^{-\frac{x-\theta}{\sigma}}+1\right)}{\left(1-e^{-\frac{x-\theta}{\sigma}}\right)}\right)}$$

$$\frac{d^2ln}{d\alpha^2} = \frac{2}{\left(2\alpha-e^{\frac{x-\theta}{\sigma}}-1\right)\ln\left(\alpha - \frac{(\alpha-1)\left(e^{-\frac{x-\theta}{\sigma}}+1\right)}{\left(1-e^{-\frac{x-\theta}{\sigma}}\right)}\right)} \quad \ldots (2.1.6)$$

$$\frac{d^2ln}{d\theta d\sigma} = \frac{2(\alpha-1)e^{\frac{x-\theta}{\sigma}}\left((\sigma-x+\theta)e^{\frac{2(x-\theta)}{\sigma}} - 2\sigma\alpha e^{\frac{x-\theta}{\sigma}} + (2\alpha-1)x - 2\theta\alpha + \theta\right)\ln\left(\alpha - \frac{(\alpha-1)\left(e^{-\frac{x-\theta}{\sigma}}+1\right)}{\left(1-e^{-\frac{x-\theta}{\sigma}}\right)}\right) + ((2-2\alpha)x+2\theta\alpha-2\theta)e^{\frac{x-\theta}{\sigma}}}{\sigma^3\left(e^{\frac{x-\theta}{\sigma}}-1\right)^2\left(e^{\frac{x-\theta}{\sigma}}-2\alpha+1\right)^2\ln^2\left(\alpha - \frac{(\alpha-1)\left(e^{-\frac{x-\theta}{\sigma}}+1\right)}{\left(1-e^{-\frac{x-\theta}{\sigma}}\right)}\right)} \quad \ldots (2.1.7)$$

$$\frac{d^2 ln}{d\theta d\alpha} = \frac{2e^{\frac{x-\theta}{\sigma}}\left(\left(e^{-\frac{x-\theta}{\sigma}}-1\right) ln\left(\alpha-\frac{(\alpha-1)\left(e^{-\frac{x-\theta}{\sigma}}+1\right)}{\left(1-e^{-\frac{x-\theta}{\sigma}}\right)}\right)+2\alpha-2\right)}{\sigma\left(e^{\frac{x-\theta}{\sigma}}-1\right)\left(2\alpha-e^{\frac{x-\theta}{\sigma}}-1\right)\ln\left(\alpha-\frac{(\alpha-1)\left(e^{-\frac{x-\theta}{\sigma}}+1\right)}{\left(1-e^{-\frac{x-\theta}{\sigma}}\right)}\right)} \quad \dots (2.1.8)$$

$$\frac{d^2 ln}{d\sigma d\alpha} = \frac{2(x-\theta)e^{\frac{x-\theta}{\sigma}}\left(\left(e^{-\frac{x-\theta}{\sigma}}-1\right) ln\left(\alpha-\frac{(\alpha-1)\left(e^{-\frac{x-\theta}{\sigma}}+1\right)}{\left(1-e^{-\frac{x-\theta}{\sigma}}\right)}\right)+2\alpha-2\right)}{\sigma^2\left(e^{\frac{x-\theta}{\sigma}}-1\right)\left(2\alpha-e^{\frac{x-\theta}{\sigma}}-1\right)\ln^2\left(\alpha-\frac{(\alpha-1)\left(e^{-\frac{x-\theta}{\sigma}}+1\right)}{\left(1-e^{-\frac{x-\theta}{\sigma}}\right)}\right)} \quad \dots (2.1.9)$$

Apparently, there is no closed form solution in (θ, σ, α). We have to use a numerical technique such as Newton- Raphson iterative procedure, to obtain the solution.

ASYMPTOTIC CONFIDENCE BOUNDS

Here we derive the asymptotic confidence bounds for unknown parameters Location (θ), Scale (σ) and Shape (α) when $\theta > 0, \sigma > 0$ and $\alpha > 0$ the simplest large sample approach is to assume that the MLEs (θ, σ, α) are approximately normal with mean (θ, σ, α) and covariance matrix I_0^{-1}, where I_0^{-1} is the inverse of the observed information matrix which defined as follows

$$I_0^{-1} = \begin{bmatrix} -E(\frac{d^2 ln}{d\theta^2}) & -E(\frac{d^2 ln}{d\theta d\sigma}) & -E(\frac{d^2 ln}{d\theta d\alpha}) \\ -E(\frac{d^2 ln}{d\theta d\sigma}) & -E(\frac{d^2 ln}{d\sigma^2}) & -E(\frac{d^2 ln}{d\alpha d\sigma}) \\ -E(\frac{d^2 ln}{d\theta d\alpha}) & -E(\frac{d^2 ln}{d\alpha d\sigma}) & -E(\frac{d^2 ln}{d\alpha^2}) \end{bmatrix}$$

The Asymptotic (1-r)100% Confident intervals for estimated parameters are as follows

$$\hat{\theta} + z_{\frac{r}{2}}[var(\hat{\theta})]$$
$$\hat{\sigma} + z_{\frac{r}{2}}[var(\hat{\sigma})],$$
$$\hat{\alpha} + z_{\frac{r}{2}}[var(\hat{\alpha})]$$

CONCLUSION

1. In this section, we introduce the three-parameter Alpha Logarithm Transformed Semi-Logistic (ALTSL) distribution. Some features of the ALTSL distribution are derived, such as the same time and design functions. We show that the new classification is flexible for both real and simulated datasets.
2. We also found that the variance, standard deviation, mean absolute difference, mean squared error, and relative error decreased as the sample size of the simulated data increased.
3. The difference is very small compared to the actual quantile values of the Alpha log-transformed semi-logistic distribution and the observed quantile values of the Alpha-log-transformed semi-logistic distribution in QQ-Plot. That's why our distribution alignment is presented well and clearly in this section.

4. In large samples, the estimator using the maximum likelihood estimation method is more effective compared to small samples.

SIMULATION STUDY

In this section, we conduct a simulation study. The main purpose of these experiments is to evaluate the effectiveness of the maximum likelihood estimation method for ALTSL distribution parameters. Use the following procedure:

Step 1: Set the sample size n and the vector of parameter value vector $\Psi = (\theta, \sigma, \alpha)$.

Step 2: Using the results obtained in step (2), calculate $(\hat{\theta}, \hat{\sigma}, \hat{\alpha})$ by the Maximum Likelihood estimation method.

Step3: Repeat steps (2) and (3) N times

Step4: Using $\widehat{\Psi}$ of Ψ, compute the Average Estimate (AE), Variance (VAR), Standard Deviation (SD), Mean Square Error (MSE), Relative Absolute Bias (RAB) and Relative Error (RE). If $\widehat{\Psi_{lm}}$ is Maximum likelihood Method estimate of $\widehat{\Psi}_m$, m=1, 2 and 3, where $\Psi_m$is a general notation that can be replaced by $\Psi_1 = \theta, \Psi_2 = \sigma\ and\ \Psi_3 = \alpha$ based on sample l, (l =1,2,…,r), then the Average Estimate (AE), Variance (VAR), Standard Deviation (SD), Mean Absolute Deviation (MAD), Mean Square Error (MSE) and Relative Absolute Bias (RAB) and Relative Error (RE) are given by the following formulas,

Average Estimate $(\hat{\psi}_m) = \frac{\sum_{i=1}^{r} \hat{\psi}_{lm}}{r}$

Variance$(\hat{\psi}_m) = \frac{\sum_{i=1}^{r} (\hat{\psi}_{lm} - \overline{\hat{\psi}_{lm}})^2}{r}$

SD $(\hat{\psi}_m = \sqrt{\frac{\sum_{i=1}^{r} (\hat{\psi}_{lm} - \overline{\hat{\psi}_{lm}})^2}{r}}$

Mean Absolute Deviation$(\hat{\psi}_m) = \frac{\sum_{i=1}^{r} Med(\left|\hat{\psi}_{lm} - \overline{\hat{\psi}_{lm}}\right|)}{r}$

Mean Square Error $(\hat{\psi}_m) = \frac{\sum_{i=1}^{r} (\widehat{\psi}_{lm} - \psi_m)^2}{r}$

Relative Absolute Bias$(\hat{\psi}_m) = \frac{\sum_{i=1}^{r} \left|(\widehat{\psi}_{lm} - \psi_m)\right|}{r\psi_m}$

$$\text{Relative Error}(\hat{\psi}_m) = \frac{1}{r}\left(\frac{\sum_{i=1}^{r} MSE\sqrt{(\widehat{\psi}_{lm})}}{\psi_m}\right)^2$$

Results were calculated using R (R Core Development Team) software. The seed used to generate a random value. This process is done to determine N=10,000 selection value and n=(20, 40, 60, 80 and 100,250) population parameter values.

APPLICATIONS

In this section, we consider two real documents. To examine the character of the new distribution, real-life time-to-event data from baseball tournaments played between 1986 and 2021, including survival times of patients in the end-stage affecting ALTSL leukemia,.

Our simulation study in Section 4.4 shows that the ML estimator should be used to estimate the parameters of the ALTSL distribution. Initially, we compared the predictions obtained from different methods with the ML estimator. We then compare the results obtained for the ALTSL distribution compatible with ML estimation with some models in life, such as Half-logistic, logistic, Gamma, lognormal distribution, Weibull and general distribution. The Kolmogorov-Smirnov (KS) test is considered to check the goodness of fit.

This procedure is based on the KS statistic $D_n = sup_x|F_n(x) - F(x;\theta,\sigma,\alpha)|$

Where sup_x means the maximum of the distance,
$F_n(x)$ is the empirical distribution function and $F(x;\theta,\sigma,\alpha)$ is cumulative distribution function of ALTSL.

In this case, we test the null hypothesis that the data comes from $F(x;\theta,\sigma,\alpha)$, and, with significance level of 5%, we will reject the null hypothesis if p value is smaller than 0.05. As discrimination criterion method, we considered the AIC (Akaike Information Criteria) computed, respectively, by

AIC = $-2l\left(\widehat{\Psi},x\right) + 2k$

Where 'k' is the number of parameters fitted and $\widehat{\Psi}$ is estimate of Ψ.

Data analysis in sport: Time-to-event data:

In this section, time data of different basketball games are obtained and ALTSL classification is explained. These games were played between 1986-2021. The observations in this file represent the waiting time for the first goal. Grand mean (SM) of time-to-event data: 0.23, 0.261, 0.87, 0.210, 0.23, 0.47, 0.52, 0.25, 0.47, 5, 12, 0, 0.89, 0.51, 0.603, 8, 3, 0 16, 9.52, 15.6, 11.2, 5.4, 8, 6.3, 8.4, 8, 5, 3.4 and 9.8.We obtained

$\hat{\theta} = 0.134\,, \hat{\sigma} = 4.3262\; and\; \hat{\alpha} = 6.532$ Results of the KS test (p value), AIC for the different probability distributions considering the above data set

Test	ALTSL	Half Logistic	Logistic	Gamma	Log normal	Weibull	Generalised Exponential
KS	0.8324	0.7056	0.6652	0.2008	0.1471	0.5336	0.4368
AIC	1935.0326	2596.50	2955.32	14856.06	9653.65	3568.96	6983.69

Data Set 2

The survival times of the patients affected the Leukemia at the final stage 0.2, 0.3, 0.3, 6, 8, 14, 1.8, 2.1, 2.5, 3.4, 4.0, 4.3, 4.8, 6.5, 5, 8, 9, 1, 3, 2, 2.1, 2.6, 3.2, 4.2, 8, 6, 10, 13, 1.8 and 4.6.
We obtained

$\hat{\theta} = 0.2963\,, \hat{\sigma} = 6.3296\; and\; \hat{\alpha} = 9.6321$ Results of the KS test (p value), AIC for the different probability distributions considering the above data set

Test	ALTSL	Semi Logistic	Logistic	Gamma	Log normal	Weibull	Generalised Exponential
KS	0.9632	0.9106	0.7652	0.2008	0.1471	0.6395	0.5237
AIC	2134.65	2763.96	3189.35	19653.52	13417.693	5563.78	11638.52

Compared to the empirical study transformation, the ALTSL distribution was found to be better for the selected model. AIC confirms this result as the ALTSL distribution has the smallest value in the selected sample. Additionally, considering the 5% significance level, the ALTSL distribution is the only model where the KS test yields a p value greater than 0.05.

OBSERVATIONS FOR THE SIMULATION RESULT

1. The maximum estimation of position (θ), scale (σ), and shape (α) are less independent.
2. The Average estimate (AE), Variance (VAR), Standard Deviation (SD), Mean Square Error (MSE), Relative Absolute Bias (RAB), Relative Error (RE) of the estimate depend on the sample size.
3. It can be stated here that the maximum prediction is obtained from the entire model. Therefore, using the optimal method will lead to better results, especially when the sample size is large. This is an interesting application of the maximum method.
4. The Average estimate (AE), Variance (VAR), Standard deviation (SD), Mean Square Error (MSE), Relative Absolute Error (RAB), and Relative Error (RE) of the estimators are independent on the population parameter values.
5. The Average estimate (AE) of the Maximum likelihood location ($\hat{\theta}$) scale ($\hat{\sigma}$) and shape ($\hat{\alpha}$) estimators are increased when sample size increased.
6. The Variance (VAR) of Maximum likelihood location ($\hat{\theta}$) scale ($\hat{\sigma}$) and shape ($\hat{\alpha}$) estimators are decreased when sample size increased.
7. The Standard Deviation of Maximum likelihood location ($\hat{\theta}$) scale ($\hat{\sigma}$) and shape ($\hat{\alpha}$) estimators are decreased when sample size increased.
8. The Mean square error (MSE) Maximum likelihood location ($\hat{\theta}$) scale ($\hat{\sigma}$) and shape ($\hat{\alpha}$) estimators are decreased when sample size increased.
9. The Relative absolute bias (RAB) Maximum likelihood location ($\hat{\theta}$) scale ($\hat{\sigma}$) and shape ($\hat{\alpha}$) estimators are decreased when sample size increased.
10. The Relative error (RE) Maximum likelihood location ($\hat{\theta}$) scale ($\hat{\sigma}$) and shape ($\hat{\alpha}$) estimators are decreased when sample size increased.

Maximum Likelihood process for estimating the ALTSL (θ, σ, α) The Newton-Raphson iteration method was used for both parameter combinations and the process was repeated 10,000 times for several samples of 100 and 250 with n = 20(20). The MLEs and their Average Estimate (AE), Variance (VAR), Mean Absolute Deviation (MAD), Mean Square Error (MSE) and Relative Absolute Bias (RAB), Relative Error (RE) of the location, scale and shape parameters are unknown population parameters of ALTSL distribution in Table 4.1.

Table 4.1 Maximum Likelihood method for estimating the ALTSL (θ= 2.5; σ=1.5; α = 2.5)

Sample size	Para meters	AE	VAR	MAD	MSE	RAB	RE
20	θ	0.2625	0.2720	0.8961	0.4425	0.0954	0.1954
	σ	1.0450	0.1846	0.7785	0.4615	0.0976	0.1976
	α	2.1019	0.2670	0.7959	0.4327	0.0942	0.1909
40	θ	0.3327	0.2659	0.7803	0.4304	0.0940	0.1940
	σ	1.0593	0.2341	0.7745	0.3680	0.0866	0.1868
	α	2.1065	0.1902	0.7567	0.2816	0.0763	0.1713
60	θ	0.3563	0.2608	0.7532	0.4204	0.0928	0.1928
	σ	1.0641	0.1643	0.6636	0.3855	0.0886	0.1886
	α	2.1151	0.1669	0.5808	0.2358	0.0709	0.1694

80	θ	0.3882	0.2104	0.6869	0.3214	0.0810	0.1810
	σ	1.0706	0.1553	0.5623	0.2130	0.0682	0.1689
	α	2.1781	0.1478	0.4347	0.1982	0.0664	0.1678
100	θ	0.4330	0.2051	0.4347	0.3109	0.0798	0.1798
	σ	1.0797	0.1715	0.5101	0.3680	0.0866	0.1866
	α	2.1811	0.1531	0.5589	0.2087	0.0677	0.1682
150	θ	0.5050	0.2041	0.3988	0.3090	0.0796	0.1796
	σ	1.0943	0.1442	0.4304	0.1911	0.0656	0.1674
	α	2.2658	0.1258	0.3379	0.1550	0.0613	0.1623
250	θ	0.5423	0.1686	0.3699	0.2391	0.0713	0.1713
	σ	1.1019	0.2024	0.4289	0.3056	0.0792	0.1792
	α	2.2680	0.1150	0.3186	0.1338	0.0588	0.1590

Maximum Likelihood process for estimating the ALTSL (θ, σ, α) The Newton-Raphson iteration method was used for both parameter combinations and the process was repeated 10,000 times for several samples of 100 and 250 with n = 20(20). The MLEs and their Average Estimate (AE), Variance (VAR), Mean Absolute Deviation (MAD), Mean Square Error (MSE) and Relative Absolute Bias (RAB), Relative Error (RE) of the location, scale and shape parameters are unknown population parameters of ALTSL distribution in Table 4.2.

Table 4.2 Maximum Likelihood method for estimating the ALTSL (θ= 0.5; σ=0.5; α = 0.5)

Sample size	Para meters	AE	VAR	MAD	MSE	RAB	RE
20	θ	0.1625	0.1846	0.7785	0.4615	0.0976	0.1976
	σ	0.1670	0.1674	0.3536	0.2367	0.0710	0.1710
	α	0.1629	0.2529	0.8297	0.4049	0.0909	0.1972
40	θ	0.1737	0.1643	0.6636	0.3855	0.0886	0.1886
	σ	0.1778	0.1409	0.2891	0.1846	0.0648	0.1648
	α	0.1724	0.2353	0.6946	0.3704	0.0868	0.1968
60	θ	0.2271	0.1715	0.5101	0.3680	0.0866	0.1866
	σ	0.2289	0.1686	0.6636	0.2391	0.0713	0.1941
	α	0.2179	0.1605	0.5528	0.2231	0.0694	0.1895
80	θ	0.2296	0.2024	0.4289	0.3056	0.0792	0.1792
	σ	0.2313	0.1354	0.2356	0.1737	0.0635	0.1635
	α	0.2200	0.1582	0.5165	0.2187	0.0689	0.1769
100	θ	0.3014	0.1794	0.3882	0.2852	0.0767	0.1767
	σ	0.3000	0.1328	0.3765	0.1688	0.0630	0.1630
	α	0.2810	0.1554	0.4905	0.2132	0.0682	0.1702
150	θ	0.3033	0.2430	0.3806	0.2705	0.0750	0.1750
	σ	0.3018	0.1181	0.2998	0.1399	0.0595	0.1595
	α	0.2827	0.1538	0.4724	0.2100	0.0678	0.1682
250	θ	0.3141	0.1440	0.3620	0.2604	0.0738	0.1738
	σ	0.3121	0.1060	0.2510	0.1161	0.0567	0.1567
	α	0.2918	0.1519	0.4447	0.2064	0.0674	0.1681

Maximum Likelihood process for estimating the ALTSL (θ, σ, α) The Newton-Raphson iteration method was used for both parameter combinations and the process was repeated 10,000 times for several samples of 100 and 250 with n = 20(20). The MLEs and their Average Estimate (AE), Variance (VAR), Mean Absolute Deviation (MAD), Mean Square Error (MSE) and Relative Absolute Bias (RAB), Relative Error (RE) of the location, scale and shape parameters are unknown population parameters of ALTSL distribution in Table 4.3.

Table 4.3 Maximum Likelihood method for estimating the ALTSL (θ= 3.5; σ=2.5; α = 3.5**)**

Sample size	Para meters	AE	VAR	MAD	MSE	RAB	RE
20	θ	3.1704	0.2996	0.7292	0.4968	0.1018	0.2018
	σ	2.1704	0.1794	0.3882	0.2852	0.0767	0.1767
	α	3.2288	0.3007	0.7334	0.4989	0.1021	0.1942
40	θ	3.1813	0.2756	0.6678	0.4495	0.0962	0.1962
	σ	2.1813	0.2430	0.3806	0.2705	0.0750	0.1750
	α	3.1126	0.2675	0.7294	0.4337	0.0943	0.1866
60	θ	3.2335	0.2235	0.4347	0.3470	0.0841	0.1841
	σ	2.2335	0.1440	0.3620	0.2604	0.0738	0.1738
	α	3.1365	0.2598	0.7023	0.4184	0.0925	0.1763
80	θ	3.2360	0.1650	0.4063	0.2320	0.0704	0.1704
	σ	2.2360	0.2341	0.3494	0.2448	0.0720	0.1720
	α	3.1654	0.2408	0.5660	0.3811	0.0881	0.1709
100	θ	3.3060	0.1613	0.3985	0.2248	0.0696	0.1696
	σ	2.3060	0.2817	0.2852	0.2308	0.0703	0.1703
	α	3.1752	0.1686	0.4518	0.2391	0.0713	0.1682
150	θ	3.3079	0.1470	0.3886	0.1967	0.0663	0.1663
	σ	2.3079	0.1920	0.2476	0.1907	0.0655	0.1655
	α	3.1934	0.1583	0.3339	0.2189	0.0689	0.1677
250	θ	3.3184	0.1378	0.3830	0.1786	0.0641	0.1641
	σ	2.3184	0.1501	0.3294	0.2027	0.0670	0.1664
	α	3.2216	0.1454	0.2517	0.1935	0.0659	0.1656

Maximum Likelihood process for estimating the ALTSL (θ, σ, α) The Newton-Raphson iteration method was used for both parameter combinations and the process was repeated 10,000 times for several samples of 100 and 250 with n = 20(20). The MLEs and their Average Estimate (AE), Variance (VAR), Mean Absolute Deviation (MAD), Mean Square Error (MSE) and Relative Absolute Bias (RAB), Relative Error (RE) of the location, scale and shape parameters are unknown population parameters of ALTSL distribution in Table 4.4.

Table 4.4 Maximum Likelihood method for estimating the ALTSL (θ= 1.5; σ=3.5; α = 1.5)

Sample size	Para meters	AE	VAR	MAD	MSE	RAB	RE
20	θ	1.0126	0.3007	0.7334	0.4989	0.1021	0.1942
	σ	3.0126	0.2535	0.8961	0.8927	0.0972	0.2000
	α	1.0126	0.1452	0.3536	0.4313	0.0681	0.1621
40	θ	1.0426	0.2675	0.7294	0.4337	0.0943	0.1866
	σ	3.0426	0.2441	0.7803	0.8662	0.0968	0.1951
	α	1.0426	0.1424	0.2891	0.4146	0.0650	0.1612
60	θ	1.0791	0.2598	0.7023	0.4184	0.0925	0.1763
	σ	3.0791	0.2331	0.7532	0.7515	0.0941	0.1880
	α	1.0791	0.1300	0.4198	0.1632	0.0623	0.1650

80	θ	1.0914	0.2408	0.5660	0.3811	0.0881	0.1709
	σ	3.0914	0.2318	0.6869	0.7292	0.0895	0.1755
	α	1.0914	0.1160	0.3486	0.1357	0.0590	0.1586
100	θ	1.1142	0.1686	0.4518	0.2391	0.0713	0.1682
	σ	3.1142	0.2017	0.4347	0.6972	0.0769	0.1717
	α	1.1142	0.1121	0.2356	0.3874	0.0586	0.1577
150	θ	1.1498	0.1583	0.3339	0.2189	0.0689	0.1677
	σ	3.1498	0.1808	0.3988	0.4563	0.0702	0.1683
	α	1.1498	0.1055	0.1138	0.1110	0.0728	0.0892
250	θ	1.1588	0.1501	0.3294	0.2027	0.0670	0.1664
	σ	3.1588	0.1670	0.3699	0.4347	0.0682	0.1670
	α	1.1588	0.1477	0.1596	0.1555	0.1005	0.1241

Maximum Likelihood process for estimating the ALTSL (θ, σ, α) The Newton-Raphson iteration method was used for both parameter combinations and the process was repeated 10,000 times for several samples of 100 and 250 with n = 20(20). The MLEs and their Average Estimate (AE), Variance (VAR), Mean Absolute Deviation (MAD), Mean Square Error (MSE) and Relative Absolute Bias (RAB), Relative Error (RE) of the location, scale and shape parameters are unknown population parameters of ALTSL distribution in Table 4.5.

Table 4.5 Maximum Likelihood method for estimating the ALTSL (θ= 4.5; σ=5.5; α = 4.5)

Sample size	Para meters	AE	VAR	MAD	MSE	RAB	RE
20	θ	4.0126	0.2670	0.7959	0.4327	0.0942	0.1909
	σ	5.0276	0.6448	0.8491	0.7504	0.2504	0.3489
	α	4.0276	0.3379	0.7155	0.5267	0.2383	0.3379
40	θ	4.0426	0.2341	0.7745	0.3680	0.0866	0.1868
	σ	5.1028	0.6333	0.8332	0.7362	0.2471	0.3435
	α	4.1028	0.3293	0.6939	0.5116	0.2300	0.3293
60	θ	4.0791	0.1902	0.7567	0.2816	0.0763	0.1713
	σ	5.1098	0.6302	0.8289	0.7324	0.2463	0.3420
	α	4.1098	0.3021	0.6255	0.4638	0.2037	0.3021
80	θ	4.0914	0.1669	0.5808	0.2358	0.0709	0.1694
	σ	5.1638	0.6227	0.8186	0.7232	0.2441	0.3385
	α	4.1638	0.2216	0.4228	0.3222	0.1258	0.2216
100	θ	4.1142	0.1553	0.5623	0.2130	0.0682	0.1689
	σ	5.1980	0.6115	0.8031	0.7095	0.2410	0.3332
	α	4.1980	0.2196	0.4177	0.3187	0.1239	0.2196
150	θ	4.1498	0.1531	0.5589	0.2087	0.0677	0.1682
	σ	5.2499	0.6128	0.8049	0.7111	0.2413	0.3339
	α	4.2499	0.1918	0.3479	0.2699	0.0970	0.1918
250	θ	4.1588	0.1478	0.4347	0.1982	0.0664	0.1678
	σ	5.3246	0.5896	0.7727	0.6825	0.2348	0.3229
	α	4.3246	0.1717	0.1939	0.1685	0.1165	0.1262

Maximum Likelihood process for estimating the ALTSL (θ, σ, α) The Newton-Raphson iteration method was used for both parameter combinations and the process was repeated 10,000 times for several samples of 100 and 250 with n = 20(20). The MLEs and their Average Estimate (AE), Variance (VAR), Mean Absolute Deviation (MAD), Mean Square Error (MSE) and Relative Absolute Bias (RAB), Relative Error (RE) of the location, scale and shape parameters are unknown population parameters of ALTSL distribution in Table 4.6

Table 4.6 Maximum Likelihood method for estimating the ALTSL (θ= 5.5; σ=4.5; α = 5.5)

Sample size	Para meters	AE	VAR	MAD	MSE	RAB	RE
20	θ	5.0276	0.2529	0.8297	0.4049	0.0909	0.1972
	σ	4.0126	0.5761	0.7541	0.6660	0.2310	0.3166
	α	5.1325	0.2835	0.3486	0.3060	0.1481	0.1788
40	θ	5.1028	0.2353	0.6946	0.3704	0.0868	0.1968
	σ	4.0426	0.5431	0.7083	0.6253	0.2216	0.3010
	α	5.1432	0.2279	0.2717	0.2376	0.1324	0.1526
60	θ	5.1098	0.1686	0.6636	0.2391	0.0713	0.1941
	σ	4.0791	0.5377	0.7009	0.6187	0.2201	0.2985
	α	5.2036	0.2263	0.2694	0.2356	0.1320	0.1518
80	θ	5.1638	0.1605	0.5528	0.2231	0.0694	0.1895
	σ	4.0914	0.5097	0.6620	0.5842	0.2122	0.2853
	α	5.2137	0.2213	0.2626	0.2295	0.1306	0.1495
100	θ	5.1980	0.1582	0.5165	0.2187	0.0689	0.1769
	σ	4.1142	0.4831	0.6252	0.5515	0.2046	0.2728
	α	5.2693	0.2191	0.2594	0.2267	0.1299	0.1484
150	θ	5.2499	0.1554	0.4905	0.2132	0.0682	0.1702
	σ	4.1498	0.4800	0.6210	0.5477	0.2038	0.2713
	α	5.2464	0.2068	0.2425	0.2117	0.1265	0.1427
250	θ	5.3246	0.1538	0.4724	0.2100	0.0678	0.1682
	σ	4.1588	0.4492	0.5782	0.5098	0.1950	0.2568
	α	5.2531	0.1914	0.2211	0.1927	0.1221	0.1354

Maximum Likelihood process for estimating the ALTSL (θ, σ, α) The Newton-Raphson iteration method was used for both parameter combinations and the process was repeated 10,000 times for several samples of 100 and 250 with n = 20(20). The MLEs and their Average Estimate (AE), Variance (VAR), Mean Absolute Deviation (MAD), Mean Square Error (MSE) and Relative Absolute Bias (RAB), Relative Error (RE) of the location, scale and shape parameters are unknown population parameters of ALTSL distribution in Table 4.7.

Table 4.7 Maximum Likelihood method for estimating the ALTSL (θ= 6.5; σ=7.5; α = 6.5)

Sample size	Para meters	AE	VAR	MAD	MSE	RAB	RE
20	θ	6.0126	0.4409	0.5667	0.4996	0.1927	0.2529
	σ	7.0126	0.1540	0.1694	0.1468	0.1115	0.1178
	α	6.0276	0.2405	0.2909	0.2503	0.2014	0.2126
40	θ	6.0426	0.4393	0.5646	0.4977	0.1923	0.2522
	σ	7.0426	0.1524	0.1840	0.1446	0.1193	0.1197
	α	6.1028	0.2233	0.2898	0.2319	0.1776	0.1986
60	θ	6.0791	0.4361	0.5601	0.4937	0.1913	0.2506
	σ	7.0791	0.1257	0.1507	0.1157	0.1102	0.1094

	α	6.1098	0.1860	0.2545	0.1920	0.1261	0.1684
80	θ	6.0914	0.4193	0.5368	0.4730	0.1866	0.2427
	σ	7.0914	0.1248	0.2105	0.1265	0.0415	0.1187
	α	6.1638	0.1443	0.2538	0.1473	0.0683	0.1344
100	θ	6.1142	0.3588	0.4530	0.3986	0.1695	0.2142
	σ	7.1142	0.1213	0.1822	0.1227	0.0365	0.1158
	α	6.1980	0.1417	0.2377	0.1445	0.0647	0.1323
150	θ	6.1498	0.3151	0.3925	0.3449	0.1571	0.1937
	σ	7.1498	0.1169	0.1397	0.1101	0.1072	0.1059
	α	6.2499	0.1314	0.2376	0.1335	0.0506	0.1240
250	θ	6.1588	0.2948	0.3644	0.3200	0.1514	0.1841
	σ	7.1588	0.1108	0.1500	0.1114	0.0220	0.1073
	α	6.3246	0.1021	0.1358	0.1021	0.0101	0.1002

Maximum Likelihood process for estimating the ALTSL (θ, σ, α) The Newton-Raphson iteration method was used for both parameter combinations and the process was repeated 10,000 times for several samples of 100 and 250 with n = 20(20). The MLEs and their Average Estimate (AE), Variance (VAR), Mean Absolute Deviation (MAD), Mean Square Error (MSE) and Relative Absolute Bias (RAB), Relative Error (RE) of the location, scale and shape parameters are unknown population parameters of ALTSL distribution in Table 4.8

Table 4.8 Maximum Likelihood method for estimating the ALTSL (θ= 5.5; σ=4.5; α = 5.5)

Sample size	Para meters	AE	VAR	MAD	MSE	RAB	RE
20	θ	7.0228	0.1877	0.2783	0.2278	0.1488	0.1045
	σ	6.0386	0.3248	0.5244	0.4245	0.2250	0.2521
	α	7.0276	0.4234	0.5230	0.3219	0.2119	0.2246
40	θ	7.0601	0.1819	0.3229	0.2524	0.0874	0.1819
	σ	6.0756	0.2584	0.4052	0.3293	0.1881	0.1806
	α	7.1028	0.4074	0.5029	0.2584	0.2064	0.2183
60	θ	7.0703	0.1579	0.2248	0.1850	0.1323	0.0724
	σ	6.1203	0.2561	0.4012	0.3260	0.1869	0.1782
	α	7.1098	0.3564	0.4391	0.2308	0.1890	0.1986
80	θ	7.1224	0.1349	0.2047	0.1698	0.0420	0.1349
	σ	6.1315	0.2469	0.3846	0.3128	0.1817	0.1682
	α	7.1638	0.2773	0.3403	0.2003	0.1620	0.1680
100	θ	7.1354	0.1227	0.1616	0.1344	0.1127	0.0344
	σ	6.1515	0.2386	0.3697	0.3008	0.1771	0.1593
	α	7.1980	0.2480	0.3036	0.1946	0.1520	0.1567
150	θ	7.1719	0.1107	0.1437	0.1272	0.0185	0.1107
	σ	6.1763	0.1814	0.2669	0.2186	0.1453	0.0976
	α	7.2499	0.2050	0.2499	0.1838	0.1373	0.1400
250	θ	7.2121	0.1067	0.1335	0.1201	0.0146	0.1067
	σ	6.3508	0.1066	0.1327	0.1113	0.1038	0.0171
	α	7.3246	0.1648	0.1995	0.1729	0.1235	0.1245

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