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RESEARCH ARTICLE

ALGEBRAIC SOLUTION OF FERMAT'S THEOREM (MATHEMATICS, NUMBER THEORY)

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ABSTRACT

Fermat's Last Theorem (or Fermat's last theorem) is one of the most popular theorems in mathematics. Formulated in French mathematician Pierre Fermat in 1637. Despite the simplicity of the formulation, literally, at the “school” arithmetic level, proof of the theorem sought by many mathematicians for more than three hundred years. And only in 1994 year the theorem was proven by the English mathematician Andrew Wilson with colleagues; The proof was published in 1995. [1]-[5] The author of this article has been searching for his own for a long time. accessible algebraic solution to this problem and believes that he succeeded,which he presents in this article.

Keywords: Theorem, Fermat, elementary, solution

INTRODUCTION

$$X^n + Y^n = Z^n \tag{01}$$

where: n- prime number, n>2; X,Y,Z are integers.

The solutions of which can be X, Y, Z - relatively prime numbers.

1.Decomposition of (01) into multipliers.

If n is odd, then (01) will decompose into multipliers:

$$X^n + Y^n = (X + Y)(X^{n-1} - X^{n-2}Y + \ldots - XY^{n-2} + Y^{n-1}) \tag{02}$$

where in the second bracket is the geometric progression

first term $a_1 = X^{n-1}$,and a multiplier $q = -\frac{Y}{X}$

The sum of the members of which $S = \frac{a_1(1-q^n)}{1-q}$

$$Z^n = Z_{11}\, Z_{22} \tag{03}$$

where:

$$Z_{11} = X + Y \tag{04}$$

$$Z_{22} = X^{n-1} - X^{n-2}Y + \ldots - X\, Y^{n-1} + Y^{n-1} \tag{05}$$

2.Equivalent representation Z_{22} .

If we sum the equidistant terms from the middle term of the progression Z_{22} in pairs.

of the middle term of the progression in pairs we have:

for degree 3

$$Z_{22} = (X+Y)^2 - 3XY \tag{06}$$

Fifth degree :

$$Z_{22} = \frac{X^5 + Y^5}{X+Y} = X^4 - X^3 Y + X^2 Y^2 - X\, Y^3 + Y^4 \tag{07}$$

$$X^4 + Y^4 = (X+Y)^4 - 4\, X\, Y\, (X+Y)^2 + 2\, X^2\, Y^2 \tag{08}$$

$$-X\, Y^3 - X^3\, Y = -X\, Y\, (X^2 + Y^2) = -X\, Y (X+Y)^2 + 2\, X^2\, Y^2 \tag{09}$$

$$Z_{225} = (X+Y)^4 - 5(X+Y)^2 + 5\, X^2\, Y^2 \tag{10}$$

to the 7th degree:

$$Z_{227} = (X+Y)^6 - 7\, X\, Y\, (X+Y)^4 + 14\, X^2\, Y^2 (X+Y)^2 - 7\, X^3\, Y^3 \tag{11}$$

degree n:

$$Z_{22N} = \frac{X^n + Y^n}{X+Y} =$$

$$= (X+Y)^{n-1} - K_{n-3} XY(X+Y)^{n-3} + \ldots \mp K_2\, X^{\frac{n-3}{2}}\, Y^{\frac{n-3}{2}} (X+Y)^2 \pm nX^{\frac{n-1}{2}}\, Y^{\frac{n-1}{2}} \tag{12}$$

$$Z_{22N} = (X+Y)^{n-1} - K_{n-3}XY(X+Y)^{n-3} + \ldots \mp K_2 X^{\frac{n-3}{2}} Y^{\frac{n-3}{2}} (X+Y)^2 \pm nX^{\frac{n-1}{2}} Y^{\frac{n-1}{2}} \qquad (*)$$

where $K_{n-3} \ldots K_2$ corresponding coefficients at $(XY)^{\cdots}(X+Y)^{\cdots}$

equivalent representation Z_{22N} algebraic sum of even powers of

X+Y and the residual term $\pm n\, X^{\frac{n-1}{2}} Y^{\frac{n-1}{2}}$.

Lemma1: Suppose that for an n odd number and for the previous n-2

an equivalent representation(*) is valid, then for the next n+2 it is

(*) is valid.

We show the transition from the two previous odd degrees to the next one

further:

$$Z_n^n = X^n + Y^n \qquad (14a)$$

$$(X^{n-2} + Y^{n-2})(X^2 + Y^2) = X^n + Y^n + Y^2 X^{n-2} + Y^2 Y^{n-2} \qquad (14b)$$

$$Z_n^n = X^n + Y^n = (X^{n-2} + Y^{n-2})(X^2 + Y^2) - X^2 Y^2 (X^{n-4} + Y^{n-4}) \qquad (14c)$$

details:

$$\frac{X^{n_1} + Y^{n_1}}{X+Y}$$

multiply by $(X+Y)^2 - 2XY$

$$X^{n+2} + Y^{n+2} = (X+Y)[Z_{22N}(X+Y)^2 - 2XYZ_{22N} - X^2 Y^2 (X^{n-2} + Y^{n-2})] \qquad (14)$$

$$(X+Y)^{n-1} - K_{n-3}XY(X+Y)^{n-3} + \ldots \mp K_2 X^{\frac{n-3}{2}} Y^{\frac{n-3}{2}} (X+Y)^2 \pm nX^{\frac{n-1}{2}} Y^{\frac{n-1}{2}} \; *$$

$$* \; (X+Y)^2 = (X+Y)^{n+1} - K_{n-1(01)}XY(X+Y)^{n-1} + \ldots \mp K_{4(01)} X^{\frac{n-3}{2}} Y^{\frac{n-3}{2}} \pm nX^{\frac{n-1}{2}} Y^{\frac{n-1}{2}} (X+Y)^2 \qquad (15)$$

$$-2XY\,[\,(X+Y)^{n-1} - K_{n-3}XY(X+Y)^{n-3} + \ldots \mp K_2 X^{\frac{n-3}{2}} Y^{\frac{n-3}{2}} (X+Y)^2 \pm nX^{\frac{n-1}{2}} Y^{\frac{n-1}{2}}\,] =$$

=

$$-2XY(X+Y)^{n-1}-K_{n-3(02)}2X^2Y^2(X+Y)^{n-3}+...\mp K_{2(02)}2X^{\frac{n-1}{2}}Y^{\frac{n-1}{2}}(X+Y)^2\pm 2X^{\frac{n+1}{2}}Y^{\frac{n+1}{2}} \quad n$$

(16)

$$-X^2Y^2(X+Y)^{n-3}+K_{n-3(03)}X^3Y^3(X+Y)^{n-5}+...\pm K_{2(03)}X^{\frac{n-1}{2}}Y^{\frac{n-1}{2}}(X+Y)^2\mp(n-2)X^{\frac{n+1}{2}}Y^{\frac{n+1}{2}} \quad (17)$$

where $K_{...(01)}$ - corresponding coefficients when multiplied by $(X+Y)^2$,

$K_{...(02)}$ - corresponding coefficients when multiplied by $-2XY$,

$K_{...(03)}$ -corresponding coefficients when multiplied by $-X^2Y^2$

After adding these algebraic terms we again obtain(*)

Theorem 1.

The equivalent representation (*) is valid for any prime n.By lemma , if the two previous representations of (*)are valid,of degree 3 and 5, then it is valid for degree 7. Now taking the previous 5 and 7 degrees we have its validity for the 9th degree, etc,which means all odd degrees are described by the above formula.And since it includes prime n , it is valid for prime n. Let us represent (1) as:

$$(X+Y)^n-Z^n=nX^{n-1}Y+\frac{n(n-1)}{2}X^{n-2}Y^2+...+\frac{n(n-1)}{2}Y^{n-2}X^2+nXY^{n-1}$$

$$(X+Y-Z)[(X+Y-Z)^{n-1}-nk_{n-3}(X+Y)Z(X+Y-Z)^{n-3}\pm nk_2(X+Y)^{n-3}Z^{n-3}(X+Y-Z)^2\mp n(XY)^{\frac{n-1}{2}}]$$

$$=nX^{n-1}Y+\frac{n(n-1)}{2}X^{n-2}Y^2+...+\frac{n(n-1)}{2}Y^{n-2}X^2+nXY^{n-1} \quad (18)$$

it follows:

$$Z_{22}=(X+Y)^{n-1}-nXY(...) \quad (19)$$

What indicates the presence of n inX+Y – Z .

.

And let's separate the common multiplier n:

$$Z_{22n}=(X+Y)^{n-1}-nk_{n-3}XY(X+Y)^{n-3}+...\pm nk_2X^{\frac{n-3}{2}}Y^{\frac{n-3}{2}}(X+Y)^2\mp nX^{\frac{n-1}{2}}Y^{\frac{n-1}{2}} \quad (20)$$

Analysis of Equation (20).

From equation (20) $Z_{11}=X+Y$ and Z_{22} cannot have a common factor for except for n. From which the following equalities follow in the absence of n:

$$X+Y=Z_1^n \quad , \quad Z-X=Y_1^n \quad , \quad Z-Y=X_1^n \tag{21}$$

$$Z_{11}=Z_1^n \, , \, Z_{22}=Z_2^n \, , \, X_{11}=X_1^n \, , \, X_{22}=X_2^n \, , \, Y_{11}=Y_1^n \, , \, Y_{22}=Y_2^n \tag{22}$$

$$X+Y-Z=n\ X_1\ Y_1\ Z_1\ K_o \tag{23}$$

where

K_o -an integer coprime to the others specified

except n.

$$Z_1^n=X_1^n+Y_1^n+2\ n\ X_1\ Y_1\ Z_1\ K_o \tag{24}$$

$$X-Y=X_1^n-Y_1^n \tag{25}$$

$$Z_1^n-Z=n\ X_1\ Y_1\ Z_1\ K_o \tag{26}$$

$$Z_2=Z_1^{n-1}-n\ X_1\ Y_1\ K_o \tag{27}$$

$$X-X_1^n=n\ X_1\ Y_1\ Z_1\ K_o \tag{28}$$

$$X_2=X_1^{n-1}+n\ Z_1\ Y_1\ K_o \tag{29}$$

$$Y-Y_1^n=n\ X_1\ Y_1\ Z_1\ K_o \tag{30}$$

$$Y_2=Y_1^{n-1}+n\ Z_1\ X_1\ K_o \tag{31}$$

$$2\ X=Z_1^n-Y_1^n+X_1^n \tag{32}$$

$$2\ Y=Z_1^n-X_1^n+Y_1^n \tag{33}$$

$$2\ Z=Z_1^n+X_1^n+Y_1^n \tag{34}$$

$$Z_1^n-X_1^n-Y_1^n=2\ n\ X_1\ Y_1\ Z_1\ K_o \tag{35}$$

$$Z_1^n-[(X_1+Y_1)^n-n\ X_1^{n-1}Y_1-\ldots-n\ Y_1^{n-1}X_1]=2\ n\ X_1\ Y_1\ Z_1\ K_o \tag{36}$$

$$Z_1-X_1-Y_1=nK_n \text{ from which it follows } Z_1>n \tag{37}$$

Note X, Y, Z are coprime numbers, as well as $X_{1,}X_{2,}Y_{1,}Y_{2,}Z_{1,}Z_2$.

If the sum or difference of two coprime numbers has a factor n, then

the sum and difference of the n-power of these numbers is divisible by at least n^2,

which is obvious from (20), (04).
If in the expansion Z, X, Y has a prime factor n

$$Z_{22}= nZ_2^n \quad , \quad X_{22}= nX_2^n \quad , \quad Y_{22}= nY_2^n \qquad (38)$$

and according to formula (20) Z_2 cannot have n available, otherwise

this will lead to the presence of it in X or Y, and vice versa, which is not acceptable.

$Z_2\ , X_2\ , Y_2$ - does not contain the factor n. In this regard, if Z contains a factor n, then formula (26) has the form, since sum $X_1^n + Y_1^n$ contains a multiplier n^m where

natural number, $m \geq 2$ and

$Z_2\ , X_2\ , Y_2$ - does not contain the factor n.

$$n^{nm-1} Z_1^n = X_1^n + Y_1^n + 2\text{n}^m\ X_1\ Y_1\ Z_1\ K_o \qquad (39)$$

To solve (34) in integers, degree n in $X_1^n + Y_1^n$,should be equal

degree n in the last monomial, that is, minimally n^2 .

similar:

$$n^{nm-1}\ X_1^n = Z_1^n - Y_1^n - 2\ n^m\ X_1\ Y_1\ Z_1\ K \qquad (40)$$

$$n^{nm-1}\ Y_1^n = Z_1^n - X_1^n - 2\ n^m\ X_1\ Y_1\ Z_1\ K \qquad (41)$$

$$n^{nm-1}\ Z_1^n - n^m\ Z_1\ Z_2\ = n^m X_1\ Y_1\ Z_1\ K \quad , \quad Z= n^m\ Z_1\ Z_2 \qquad (42)$$

$$n^m X_1\ X_2 - n^{nm-1}\ X_1^n = n^m\ X_1\ Y_1\ Z_1\ K \quad , \quad X = n^m\ X_1\ X_2 \qquad (43)$$

$$n^m\ Y_1\ Y_2 - n^{nm-1}\ Y_1^n = n^m\ X_1\ Y_1\ Z_1\ K \quad , \quad Y = n^m\ Y_1\ Y_2 \qquad (44)$$

What follows:

$$Z_2 = n^{nm-m-1}\ Z_1^{n-1} - X_1\ Y_1\ K \qquad (45)$$

$$X_2 = n^{nm-m-1}\ X_1^{n-1} + Z_1\ Y_1\ K \qquad (46)$$

$$Y_2 = n^{nm-m-1} Y_1^{n-1} + Z_1 X_1 K \quad (47)$$

If there is n in Z, we assign it to some $Z_1 = n^n Z_1^n$.

Thus

$X + Y - Z = n X_1 Y_1 Z_1 K_o$ universal,

where $X_{1,} Y_1 Z_{1,} K_o$ -coprime corresponds to X,Y,Z with and without n. The difference is

$$K_o = n^{m-1} K \quad (48)$$

Degree n=3.

According to (28) and Newton's binomial[6]:

$$Z_2^3 = Z_1^6 - 3(X_1^3 + 3 X_1 Y_1 Z_1 K_o)(Y_1^3 + 3 X_1 Y_1 Z_1 K_o) = (Z_1^2 - 3 X_1 Y_1 K_o)^3 =$$

$$= Z_1^6 - 9 Z_1^4 X_1 Y_1 K_o + 27 Z_1^2 X_1^2 Y_1^2 K_o^2 - 27 X_1^3 Y_1^3 K_o^3 \quad (49)$$

On the other side :

$$Z_2^3 = (X+Y)^2 - 3 X Y =$$

$$= Z_1^6 - 3 X_1^3 Y_1^3 - 9 X_1^3 X_1 Y_1 Z_1 K_o - 9 Y_1^3 X_1 Y_1 Z_1 K_o - 27 X_1^2 Y_1^2 Z_1^2 K_o^2 \quad (50)$$

$$-3(X_1^3 + Y_1^3 - Z_1^3 + Z_1^3) 3 X_1 Y_1 Z_1 K_o = 2*27 X_1^2 Y_1^2 Z_1^2 K_o^2 - 9 X_1 Y_1 Z_1^4 K_o \quad (51)$$

$$9K_o^3 = 1, \quad K_o^3 = \frac{1}{9} \quad (52)$$

If Z contains n:

$$X + Y = 3^{3m-1} Z_1^3 \quad (53)$$

$$3 Z_2^3 = 3(3^{3m-1} Z_1^2 - 3^m X_1 Y_1 K_o)^3 = 3^{9m-2} Z_1^6 - 3^{6m+1} Z_1^4 X_1 Y_1 K_o +$$

$$+ 3^{5m+1} Z_1^2 X_1^2 Y_1^2 K_o^2 - 3^{3m+1} X_1^3 Y_1^3 K_o^3 \quad (54)$$

$$(X+Y)^2 - 3 X Y = 3^{6m-2} Z_1^6 - 3 X Y \quad (55)$$

n – prime number

According to paragraph 3, the sum of two integers X,Y to an prime power n equal to the third number

Z to the power n only when performed necessary condition, namely (21):

$$X+Y=Z_1^n \quad (52)$$

$$Z-X=Y_1^n \quad (53)$$

$$Z-Y=X_1^n \quad (54)$$

and if one of these conditions is not met, there is no solution in integers.Fermat's theorem in integers, so there is no need to consider the case of n being in one of three numbers.

Let be:

$$X_3^3+Y_3^3=Z_3^3 \quad (55)$$

where

Z_3 -integer.

And then, since the third power has no solution in integers:

$$X_3+Y_3\neq Z_3^3 \quad (56)$$

where

X_3+Y_3 not integers, but irrational, since Fermat's theorem in the third

powers are not solvable in integers.

Let's say :

$$X^n+Y^n=Z^n \quad (57)$$

$$X+Y=Z_1^n \quad (58)$$

Multiply (55) by Z^{n-3} and we will equate $Z_3=Z$:

$$Z_1^n Z_2^n=Z^{n-3}(X_3^3+Y_3^3) \quad (59)$$

But then:

$$\frac{X_3^3+Y_3^3}{Z_1^3}=\frac{Z^3}{Z_1^3}=Z_2^3 \quad (60)$$

which means that the third power of Fermat's theorem has an integer solution.

Thus, assumption (58) is incorrect and further- inequality (57)!

CONCLUSION

If the degree in (01) is odd, there is no solution. Pharm proved the absence of a solution for the 4thdegree and thereby proved its absence for everyone $n = 2^m$,where m is an integer. Fermat' theorem is solvable in the first and second powers!

REFERENCES

1. The Moment of Proof: Mathematical Epophanies.—Oxford University Press, 1999.—ISBN 0-19-513919-4.

2. Faltings, Gerd(1995).The Proof of Fermat's last theorem by R. Taylor and A. Wiles,Notices of the AMS(42) (7), 743—746.

3. Daney, Charles(2003).The Mathematics of Fermat's last theorem. Retrieved Aug. 5, 2004.

4. O'Connor, J. J. & and Robertson, E. F.(1996).Fermat's last theorem. The history of the problem. Retrieved Aug. 5, 2004.

5. Shay, David (2003).Donald C. Benson.. Retrieved Aug. 5, 2004.

6. https://www.britannica.com/science/binomial-theorem