AJMS
Asian Journal of Mathematical Sciences

RESEARCH ARTICLE

# SPECTRAL THEORY AND FUNCTIONAL CALCULI IN THE REFLEXIVE BANACH SPACES

*** Mykola Ivanovich Yaremenko**

*Department of Partial Differential Equations,The National Technical University of Ukraine,"Igor Sikorsky Kyiv Polytechnic Institute", Kyiv, Ukraine*

**Corresponding Email: Math.kiev@gmail.com**

**Received: 28-06-2024; Revised: 26-07-2024; Accepted: 28-08-2024**

## ABSTRACT

This article establishes the correspondence between the functional calculi for the operator defined on the Banach spaces and the spectral decomposition. We show that there is a functional calculus on Borel algebra for each well-bounded operator $A \in BL(X)$, which uniquely corresponds with a multiplication operator on some $L^p(\Omega, \Theta, \mu_\Omega)$.

**Keywords:** functional calculus, spectral family, spectral theorem, C*-algebra, measurable space, spectral integral, well-bounded operator

## INTRODUCTION

The classical spectral theory plays a fundamental role in quantum physics since observables are normal operators, the eigenvalues of which represent the possible measurement event, and mixed states correspond with trace-class operators [16]. Unlike Hilbert space, in the spectral theory for Banach spaces, there are many aspects that need to be clarified, according to E. Kowalski: "the general picture for Banach spaces is barely understood today" [18].

Some aspects of the spectral theorem for well-bounded operators were developed by I. Doust, who showed the existence of a relationship between well-bounded operators and scalar-type operators, and if an operator has a contractive absolutely continuous functional calculus then this operator can be represented by the spectral integral [19]. For general reference and the history of the question see [4-7, 10-19].
In this article, we study the correlation between the functional calculi on the Banach spaces and the spectral decomposition. We prove that assume bounded operator $A: L^p \to L^p$ is well-bounded then all spectra $\sigma(A)$ of the operator $A$ is real and has representation $A = \int\limits_{[a,b]}^{\oplus} s dE(s)$. Finally, we establish that

a well-bounded of type (B) operator $A \in BL(X)$ possesses a Borel functional calculus and is uniquely equivalent to a multiplication operator on some $L^p(\Omega, \Theta, \mu_\Omega)$ .
In the case of the Hilbert space, contractive projections possess the property of orthogonality, so they are self-adjoint orthogonal projective mappings on the Hilbert space. In the $L^p$ -space case, the contractive projections have properties similar to the $L^2$ -orthogonality. So, assume a sequence $\{E_i\}$ on some $L^p$ -space such that $E_i \subseteq E_{i+1}$ and $\|E_i\| \le 1$ then $\{E_i\}$ corresponds to martingales on $L^p$ -space by the Burkholder theorem. In $L^p$ -space, the martingale is given by $\{\psi_i = E_i\psi\}$, if we denote a scalar sequence $\{\alpha_i, \ \ |\alpha_i| \le 1\}$ then the Burkholder theorem yields $\left\| \sum_{i=1,\dots,n} \alpha_i (E_i - E_{i-1})\psi \right\|_p \le \left( \max\left\{ p, (p-1)^{-1} p \right\} - 1 \right) \|\psi\|_p$ for all natural numbers $n$ and all elements $\psi \in L^p$ .

**A short summary of the Hilbert space case**

The main goal of this article is to develop spectral theory on the reflexive Banach spaces and especially consider the $L^p$ cases, so, first, we consider a well-established Hilbert space theory.

Let $H$ be a Hilbert space. Let $A: H \to H$ be a continuous linear normal operator on the Hilbert space, an operator $A: H \to H$ is normal if and only if $AA^* = A^*A$, where the operator $A^*$ is a Hermitian adjoint. Let $(\Omega, \Theta, \mu_\Omega)$ be a measure space, where $\Theta$ is a $\sigma$- algebra of $\mu_\Omega$- measurable subspaces. Then spectral theory states: for any normal operator $A: H \to H$ there is a uniquely defined Borel functional calculus $\Phi: C([a, b]) \to LB(H)$ defined on the spectrum $\sigma(A)$, which can be expressed by the equality $\Phi(\psi) = U^{-1} m_{\psi \circ \phi} U$ for some unitary operator $U: H \to L^2(\Omega, \Theta, \mu_\Omega)$, and the multiplication operator $m_\phi$ on $L^2(\Omega, \Theta, \mu_\Omega)$ given by $m_\phi \psi = \phi\psi$ for all $\psi \in dom(m_\phi) \subset L^2(\Omega, \Theta, \mu_\Omega)$.

If $A = A^*$ then the operator $A \in LB(H)$ is called self-adjoint. For self-adjoint operators, we have a stronger statement of the spectral theory: let $A: H \to H$ be a self-adjoint operator then: there is the spectral measure $E(t)$ each element of which is a self-adjoint operator and such that

$$A = \int_{[a,b]}^{\oplus} t dE(t) \quad , \qquad (1)$$

for the operator $A \in LB(H)$ equality

$$\|p(A)\| \le |p(b)| + \int_{[a,b]} |p'(s)| ds \qquad (2)$$

holds for all polynomials $p$ .

The construction of multiplication operator representation for continuous functional calculus is based on the Riesz–Markov– Kakutani representation theorem. We consider an unital *-homomorphism $\Phi : C([a,b]) \to LB(H \to H)$ where the set $[a,b]$ is compact then we can show that there exists a unitary operator $U : H \to L^2(\Omega, \Theta, \mu_\Omega)$ and an unital *-homomorphism $\Upsilon : C([a,b]) \to L^p(\Omega, \Theta, \mu_\Omega)$ so that $m_{\Upsilon(\psi)} = U\Phi(\psi)U^{-1}$ for all functions $\psi \in C([a,b])$. For each element $x \in H$, we construct a positively defined linear functional by $\psi \mapsto (\Phi(\psi)x, x)$ where $(\cdot,\cdot)$ is a scalar product in the Hilbert space $H$, then applying the Riesz–Markov– Kakutani theorem, we obtain the existence of a unique Borel measure $\mu(x)$ on $[a,b]$ dependent on an element $x \in H$ such that the linear function can be rewritten as

$$(\Phi(\psi)x, x) = \int_{[a,b]} \psi(s)\, d\mu(x,s) \quad . \tag{3}$$

Next, one can use extension arguments and cyclic subspace generated by $x \in H$; next, by the Zorn lemma Hilbert space decomposed into cyclic subspaces, the $\sigma$-algebra is inherited from Borel's sets and measure is a direct sum of measures, for details see T. Eisner and B. Farkas [8].

## Decomposion on $L^p$-spaces

Let $(\Omega, \Theta, \mu_\Omega)$ be a measure space and let $L^p(\Omega, \Theta, \mu_\Omega)$ be a Banach space of all Lebesgue p-integrable functions. We denote $p^* = \max\{p, q\}$, $p + q = pq$.

The general Hilbert case and $L^2(\Omega, \Theta, \mu_\Omega)$ special cases differ from the Banach case and $L^p(\Omega, \Theta, \mu_\Omega)$, that in $L^p(\Omega, \Theta, \mu_\Omega)$ there is no orthogonality property for contractive projections in a strict sense since $L^p(\Omega, \Theta, \mu_\Omega)$ do not possess the scalar product.

**Theorem 1.** ***Let an operator*** $A : L^p(\Omega, \Theta, \mu_\Omega) \to L^p(\Omega, \Theta, \mu_\Omega)$ ***be bounded for some*** $1 < p < \infty$***. If inequality (2) holds for all polynomials*** $p$ ***then all spectrum*** $\sigma(A)$ ***of the operator*** $A$ ***is real.***

**Proof.** The Banach space $L^p(\Omega, \Theta, \mu_\Omega)$ is reflexive and the space $L^q(\Omega, \Theta, \mu_\Omega)$, $p + q = pq$ is its dual and $E : R \to BL(L^p \to L^p)$ is a projection-valued function. We have to show that the real function $t \mapsto \langle E(t)\psi, \psi^* \rangle$ has a bound variation for all $\psi \in L^p(\Omega, \Theta, \mu_\Omega)$ and $\psi^* \in L^q(\Omega, \Theta, \mu_\Omega)$, $p + q = pq$.

Indeed, let $\Lambda = \{a = t_0 < t_1 < ... < t_n = b\}$ be a partition of the interval $[a,b] \subset R$.

For fixed elements $\psi \in L^p(\Omega, \Theta, \mu_\Omega)$, $\psi^* \in L^q(\Omega, \Theta, \mu_\Omega)$, $p + q = pq$, we consider the variation of $\langle E(t)\psi, \psi^* \rangle$ as a function of $t$, and we calculate

$$\underset{[a,b]\subset R}{\operatorname{var}}\left(\left\langle E(t)\psi,\psi^*\right\rangle\right)=\sum_{i=\mathbf{1},\dots,n}\left|\left\langle E(t_i)\psi,\psi^*\right\rangle-\left\langle E(t_{i-\mathbf{1}})\psi,\psi^*\right\rangle\right|=$$
$$=\sum_{i=\mathbf{1},\dots,n}\left|\left\langle\left(E(t_i)-E(t_{i-\mathbf{1}})\right)\psi,\psi^*\right\rangle\right|\le\left\|\sum_{i=\mathbf{1},\dots,n}\left(E(t_i)-E(t_{i-\mathbf{1}})\right)\right\|_{L^p\to L^p}\left\|\psi\right\|_{L^p}\left\|\psi^*\right\|_{L^q}.$$

All functions of bounded variation over interval $[a,b]\subset R$ constitute a Banach algebra $BV([a,b])$ with the natural norm

$$\left\|\psi\right\|_{BV([a,b])}=\left|\psi(b)\right|+\underset{[a,b]\subset R}{\operatorname{var}}(\psi),\tag{4}$$

we remark that

$$\left\|\sum_{i=\mathbf{1},\dots,n}\left(E(t_i)-E(t_{i-\mathbf{1}})\right)\right\|_{L^p\to L^p}\le\left\|\sum_{i=\mathbf{1},\dots,n-\mathbf{1}}\left(E(t_i)-E(t_{i-\mathbf{1}})\right)\right\|_{L^p\to L^p}+$$
$$+\left\|\left(E(t_n)-E(t_{n-\mathbf{1}})\right)\right\|_{L^p\to L^p}.$$

Now we show that assume $1<p<\infty$, then there exists a constant $c_p$ such that

$$\left\|\sum_{i=\mathbf{1},\dots,\dots}\alpha_i\left(E(t_i)-E(t_{i-\mathbf{1}})\right)\psi\right\|_{L^p}\le c_p\left\|\psi\right\|_{L^p}$$

for any increasing sequence of projection $E(t):LB\left(L^p(\Omega,\Theta,\mu_\Omega)\right)\to LB\left(L^p(\Omega,\Theta,\mu_\Omega)\right)$ and any numbers' sequence $\{\alpha_i\}$, and all $\psi\in L^p(\Omega,\Theta,\mu_\Omega)$.

Now, we are going to use the Burkholder-Davis-Gundy inequality in a form: for each $\psi\in L^p(\Omega,\Theta,\mu_\Omega)$, we define a martingale $\{\psi_i\}$ by the restriction $\psi_i=E(t_i)\psi$ then the martingale transform of $\psi\in L^p(\Omega,\Theta,\mu_\Omega)$ is defined by $\gamma_i=\sum_{i=\mathbf{1},\dots,n}\alpha_i(\psi_i-\psi_{i-\mathbf{1}})$ for the scalar sequence $\{\alpha_i\}$ such that $|\alpha_i|\le 1$ for all indices $i$. Applying the *Burkholder-Davis-Gundy* inequality, we obtain the estimation

$$\left\|\sum_{i=\mathbf{1},\dots,n}\alpha_i(\psi_i-\psi_{i-\mathbf{1}})\right\|_{L^p}\le\left(p^*-1\right)\left\|\psi\right\|_{L^p}$$

so that

$$\left\|\sum_{i=\mathbf{1},\dots,n}\alpha_i\left(E(t_i)-E(t_{i-\mathbf{1}})\right)\psi\right\|_{L^p}\le\left(p^*-1\right)\left\|\psi\right\|_{L^p}$$

for every natural number $n$ and all $\psi \in L^p(\Omega, \Theta, \mu_\Omega)$. For a sequence of complex numbers sequence $\{\alpha_i\}$ such that $|\alpha_i| \le 1$ we obtain the similar estimation

$$\left\| \sum_{i=1,\dots,n-1} \alpha_i \left(E(t_i) - E(t_{i-1})\right)\psi \right\|_{L^p} \le 2\left(p^* - 1\right)\|\psi\|_{L^p}.$$

Thus, we have an estimation of the operator norm

$$\left\| \sum_{i=1,\dots,n-1} \alpha_i \left(E(t_i) - E(t_{i-1})\right) \right\|_{L^p \to L^p} \le 2\left(p^* - 1\right)$$

and

$$\left\| \alpha_n \left(E(t_n) - E(t_{n-1})\right) \right\|_{L^p \to L^p} \le 3,$$

finally, we have

$$\left\| \sum_{i=1,\dots,n} \alpha_i \left(E(t_i) - E(t_{i-1})\right) \right\|_{L^p \to L^p} \le 2\left(p^* - 1\right) + 3,$$

therefore the estimation $\underset{[a,b]\subset R}{\operatorname{var}}\left(\left\langle E(t)\psi, \psi^* \right\rangle\right) \le \left(2\left(p^* - 1\right) + 3\right)\|\psi\|_{L^p} \left\|\psi^*\right\|_{L^q}$ proves the theorem.

This theorem is based on D. Burkholder theorem which states that let $L^p(\Omega, \Theta, \mu_\Omega)$ be measure space and let $\{E_i\}$ be a non-decreasing sequence of contractive projection in $L^p(\Omega, \Theta, \mu_\Omega)$ with the first $E_0 = 0$ then the estimation

$$\left\| \sum_{i=1,\dots,\dots} \alpha_i \left(E_i - E_{i-1}\right)\psi \right\|_{L^p} \le \left(p^* - 1\right)\|\psi\|_{L^p} \tag{5}$$

holds for all nonzero functions $\psi \in L^p(\Omega, \Theta, \mu_\Omega)$. So, we can formulate the following theorem as a consequence of the D. Burkholder theorem.

**Theorem 2.** ***Let*** $\{Q_i\}$ ***be an increasing sequence of projections*** $L^p(\Omega, \Theta, \mu_\Omega) \to L^p(\Omega, \Theta, \mu_\Omega)$ ***such that*** $\|Q_i\|_{L^p \to L^p} < 1$ ***and*** $\{\alpha_i\}$ ***be a scalars sequence so that*** $|\alpha_i| < 1$ ***for all indices*** $i$***. Then, the inequality***

$$\left\| \sum_{i=1,\dots,\dots} \alpha_i \left(Q_i - Q_{i-1}\right) \right\|_{L^p \to L^p} \le 2\left(p^* - 1\right) \tag{6}$$

***holds for*** $p^* = \max\{p, q\}$.

## Functional calculus on reflexive Banach spaces

Let $X$ be a separable reflexive Banach space and let space $X^*$ be dual of $X$. Since $X$ is reflexive then double dual $X^{**}$ coincides with $X$.

**Definition 1.** ***An operator-value measure*** $E(t)$ ***for the Banach space*** $X$ ***is a projection-valued function*** $E : R \to BL(X)$ ***that satisfies the following conditions:***

1) $E(t)E(s) = E(\min\{t,s\})$ ***for all*** $t, s \in R$ ***;***

2) $E(-\infty) = 0, \quad E(\infty) = I$ ***, namely,*** $\lim_{s\to-\infty} E(s) = 0$ ***and*** $\lim_{s\to\infty} E(s) = I$ ***;***

3) $\lim_{\varepsilon\to 0} E(t+\varepsilon)x = E(t)x$ ***for all*** $x \in X$ ***and all*** $t \in R$***;***

4) ***there exists a constant*** $c$ ***such that*** $\|E(t)\| \le c$ ***for all*** $t \in R$***.***

When operator-valued measure $E$ satisfies conditions $E(s) = 0$ for all $s < a \in R$ and $E(u) = I$ for all $u \ge b \in R$ then spectral measure $E$ is said to be concentrated on interval $[a,b] \subset R$.

Let $A \in BL(X)$ be well-bounded of type (B). Then, there exists a unique spectral measure $E$ concentrated on $[a,b]$ such that

$$A = \int_{[a,b]}^{\oplus} s dE(s)$$

and there exists a weak spectral measure such that equality

$$\langle Ax, x^* \rangle = -\int_{[a,b]} \langle x, E(s)x^* \rangle ds + b\langle x, x^* \rangle$$

holds for all elements $x \in X, \quad x^* \in X^*$.

Let $X$ be a reflexive strictly convex and smooth Banach space then for each fixed $x \in X$ there exists at least one element $\tilde{x}^* \in X^*$ such that $\langle x, \tilde{x}^* \rangle = \|x\|\|\tilde{x}^*\|$, this is a consequence of the Hahn-Banach theorem. For fixed $p > 1$ , we introduce a duality mapping $J_p : X \to X^*$ given by

$$J_p(x) = \left\{\tilde{x}^* \in X^* : \langle x, \tilde{x}^* \rangle = \|x\|\|\tilde{x}^*\|, \quad \|\tilde{x}^*\| = \|x\|^{p-1}\right\}$$

for all $x \in X$. The mapping $J_p$ has the following properties: $\langle x, J_p(x) \rangle = \|x\|^p$ and $J_p(x) = \|x\|^{p-2} J(x)$ for all $x \in X$.

We introduce a mapping $\tilde{J} : L(X \to X) \to L(X^* \to X^*)$ such that the equality

$$\tilde{J}(A) J_p(x) = J_p(Ax)$$

holds for all linear operators $X \to X$ and all elements $x \in X$.

**Definition 2.** ***Let*** $(\Omega, \Theta, \mu_\Omega)$ ***be Borel measure space. Let the mapping***

$$\Phi : L^1(\Omega, \Theta, \mu_\Omega) \to C(X \to X)$$

***satisfies the following conditions:***

***1)*** $\Phi(1) = I$***;***

***2)*** $\Phi(\psi) + \Phi(\varphi) \subseteq \Phi(\psi + \varphi)$ ***and***

$t\Phi(\psi) \subseteq \Phi(t\psi)$ ***for all*** $t$***;***

***3)*** $\Phi(\psi)\Phi(\varphi) \subseteq \Phi(\psi\varphi)$ ***and***

$Dom(\Phi(\psi)\Phi(\varphi)) = Dom(\Phi(\varphi)) \cap Dom(\Phi(\psi\varphi))$***;***

***4) if*** $\psi(t)$ ***is bounded, then*** $\Phi(\psi) \in LB(X \to X)$***;***

***5)*** $\Phi(\psi)$ ***is densely defined and*** $\tilde{J}(\Phi(\psi))^* = \Phi\left(\overline{\psi}|\psi|^{p-2}\right)$***;***

***6) if*** $\lim_{n\to\infty} \psi_n(t) = \psi(t)$ ***for all*** $t$***, and*** $\lim_{n\to\infty} \|\psi_n - \psi\|_\infty = 0$***, then***

$$\lim_{n\to\infty} \|\Phi(\psi_n) - \Phi(\psi)\|_{C(X\to X)} = 0$$

***for all*** $\psi, \varphi \in L^1(\Omega, \Theta, \mu_\Omega)$***. A pair*** $(\Phi, X)$ ***is called functional calculus.***

Now, let us consider a functional calculus for the multiplication operator $m_\phi \in LB(L^p \to L^p)$. We define the mapping $\Upsilon$ for the multiplication operator $m_\phi$ by $\Upsilon(\psi) = m_{\psi\circ\phi}$ for all $\phi \in L^p$. The mapping

$\Upsilon : L^1\left(essrank\left(\phi\right), \phi\circ\mu\right)\to LB\left(L^p\to L^p\right)$ is called functional calculus for multiplication operator $m_\phi\in BL\left(L^p\to L^p\right)$.

By straightforward calculation, we obtain the following properties of functional calculus for multiplication operators.

**Lemma 1.** ***Let*** $\left(\Omega, \Theta, \mu_\Omega\right)$ ***be a*** $\sigma$***-finite measurable space and*** $\phi\in L^1\left(\Omega, \Theta, \mu_\Omega\right)$ ***then the functional calculus is***

$$\Upsilon : L^1\left(essrank\left(\phi\right), \phi\circ\mu\right)\to LB\left(L^p\left(\Omega, \Theta, \mu_\Omega\right)\to L^p\left(\Omega, \Theta, \mu_\Omega\right)\right)$$

***defined by*** $\Upsilon\left(\psi\right)=m_{\psi\circ\phi}$ ***has following properties:***

***1)*** $\Upsilon\left(1\right)=I$, $\Upsilon\left(\psi\right)+\Upsilon\left(\varphi\right)\subseteq\Upsilon\left(\psi+\varphi\right)$ ***and*** $t\Upsilon\left(\psi\right)\subseteq\Upsilon\left(t\psi\right)$ ***for all*** $t$***;***

***2)*** $\Upsilon\left(\psi\right)\Upsilon\left(\varphi\right)\subseteq\Upsilon\left(\psi\varphi\right)$ ***and***

$Dom\left(\Upsilon\left(\psi\right)\Upsilon\left(\varphi\right)\right)=Dom\left(\Upsilon\left(\varphi\right)\right)\cap Dom\left(\Upsilon\left(\psi\varphi\right)\right)$***;***

***3) if and only if*** $\psi\in L^\infty\left(essrank\left(\phi\right), \phi\circ\mu\right)$***, then*** $\Upsilon\left(\psi\right)\in LB\left(L^p\left(\Omega, \Theta, \mu_\Omega\right)\to L^p\left(\Omega, \Theta, \mu_\Omega\right)\right)$***;***

***4)*** $\Upsilon\left(\psi\right)$ ***is densely defined and*** $\Upsilon\left(\psi\right)^{-1}=\Upsilon\left(\psi^{-1}\right)$ ***for all nonzero*** $\psi$***;***

***5) sequential boundness: if*** $\lim_{n\to\infty}\psi_n\left(t\right)=\psi\left(t\right)$ ***for*** $\phi\circ\mu$***-almost all*** $t$***, and*** $\lim_{n\to\infty}\left\|\psi_n-\psi\right\|_\infty=0$***, then*** $\lim_{n\to\infty}\left\|\Upsilon\left(\psi_n\right)-\Upsilon\left(\psi\right)\right\|_{L^p\left(\Omega, \Theta, \mu_\Omega\right)}=0$***.***

**Theorem 2.** ***Let the operator*** $A\in BL\left(X\right)$ ***be well-bounded of type (B). Then, the operator*** $A$ ***has a Borel functional calculus and the unique equivalence to a multiplication operator on some*** $L^p\left(\Omega, \Theta, \mu_\Omega\right)$***.***

**Proof.** For each element $x\in X$, we find an element $J_p\left(x\right)\in X^*$ such that $\left\langle x, J_p\left(x\right)\right\rangle=\left\|x\right\|^p$, and, for each function $\psi\in L^1\left(\Lambda, \Xi, \mu_\Lambda\right)$, we define a linear mapping $\psi\mapsto\left\langle\Phi\left(\psi\right)x, J_p\left(x\right)\right\rangle$. For all elements $x\in X$, we calculate

$$\begin{aligned}&\left\langle\Phi\left(\left|\psi\right|^p\right)x, J_p\left(x\right)\right\rangle=\left\langle\tilde{J}\left(\Phi\left(\psi\right)\right)^*\Phi\left(\psi\right)x, J_p\left(x\right)\right\rangle=\\&=\left\langle\Phi\left(\psi\right)x, \tilde{J}\left(\Phi\left(\psi\right)\right)J_p\left(x\right)\right\rangle=\left\langle\Phi\left(\psi\right)x, J_p\left(\Phi\left(\psi\right)x\right)\right\rangle=\\&=\left\|\Phi\left(\psi\right)x\right\|_X^p\geq 0\end{aligned}$$

therefore, the Riesz-Markov-Kakutani representation theorem renders an existence of the uniquely defined regular Borel measure $\mu_x$ dependent on $x \in X$ such that

$$\left\langle \Phi(\psi)x, J_p(x) \right\rangle = \int_{\Omega} \psi(t)\, d\mu_x(t), \tag{7}$$

we can apply the Riesz-Markov-Kakutani representation theorem since mapping $\psi \mapsto \left\langle \Phi(\psi)x, J_p(x) \right\rangle$ is positive linear functional on functional space $L^1(\Lambda, \Xi, \mu_\Lambda) \cap L^p(\Lambda, \Xi, \mu_\Lambda)$. If we take $\psi = 1$ then obtain $\|\mu_x\| = \|x\|_X^p$ for each $x \in X$ so that the equality

$$\|\Phi(\psi)x\|_X^p = \|\psi\|_{L^p(\Lambda, \mu_x)}^p \tag{8}$$

holds for all $\psi \in L^1(\Lambda, \Xi, \mu_\Lambda) \cap L^p(\Lambda, \Xi, \mu_\Lambda)$ and all $x \in X$.

For each $x \in X$, we define the mapping $T_x : L^1(\Lambda, \Xi, \mu_\Lambda) \cap L^p(\Lambda, \Xi, \mu_\Lambda) \to X$ by $\psi \mapsto \Phi(\psi)x$, mapping $T_x$ extends to an isomorphism of Banach spaces $T_x : L^p(\Lambda, \mu_x) \to \Pi(x)$ where we define a cyclic subset $\Pi(x)$ with respect to $\Phi$ by

$$\Pi(x) = clos\left\{\Phi(\psi)x : \quad \psi \in L^1(\Lambda, \Xi, \mu_\Lambda) \cap L^p(\Lambda, \Xi, \mu_\Lambda)\right\}.$$

Next, we have the following equalities

$$\begin{aligned} &\Phi(\psi)T_x\varphi = \Phi(\psi)\Phi(\varphi)x = \\ &= \Phi(\psi\varphi)x = T_x(\psi\varphi) = T_x m_\psi \varphi, \end{aligned}$$

where the multiplication operator $m_\psi$ on $L^p(\Lambda, \mu_x)$ space is given by $m_\psi \varphi \overset{def}{=} \psi\varphi$ for all $\varphi \in Dom(m_\psi) = \left\{\phi \in L^p(\Lambda, \mu_x) : \quad \psi\phi \in L^p(\Lambda, \mu_x)\right\}$. Therefore, we have $T_x m_\psi = \Phi(\psi)T_x$.

We present the space $X$ in the form of the strait sum

$$X = \bigoplus_{\alpha} \Pi(x_\alpha) = \bigoplus_{\alpha} L^p(\Lambda, \mu_{x_\alpha})$$

where $\{x_\alpha\} \subset X$, $x_\alpha$ are unit elements. We compose the set $\Lambda_\alpha = \Lambda \times \{\alpha\}$ for each $\alpha$ copy of the set $\Lambda$ and take a union $\bigcup_{\alpha} \Lambda_\alpha = \Omega$. The $\sigma$-algebra is defined by

$$\Theta = \left\{E : \quad E \cap \Lambda_\alpha \in Bor(\Lambda_\alpha) \quad \forall \alpha\right\}.$$

The measure $\mu_\Omega$ is defined as a straight sum in the form of $\mu_\Omega(E)=\sum_\alpha \mu_{x_\alpha}(E\cap\Lambda_\alpha)$ for all sets $E\in\Theta$.

Thus, we obtain the measured space $(\Omega, \Theta, \mu_\Omega)$ so Banach space $X$ decompose as

$$X=\underset{\alpha}{\oplus} L^p\left(\Lambda, \mu_{x_\alpha}\right)=L^p\left(\underset{\alpha}{\cup}\Lambda_\alpha, \underset{\alpha}{\oplus}\mu_{x_\alpha}\right)=L^p\left(\Omega, \Theta, \mu_\Omega\right).$$

We show that there exists a structure-preserving equivalence between the operator $\Phi(\psi)$ on Banach space $X$ and the multiplication operator $\Upsilon(\psi)$ on $L^p(\Omega, \Theta, \mu_\Omega)$, where the functional calculus $\Upsilon: C(\Lambda)\to L^\infty(\Omega, \Theta, \mu_\Omega)$ defined by $\Upsilon(\psi)=\psi$ on $\Lambda$.

So, we have that there exists a structure-preserving equivalence $U: X\to L^p(\Omega, \Theta, \mu_\Omega)$ and a function on $(\Omega, \Theta, \mu_\Omega)$ such that there is a multiplication operator representation in the form $A=U^{-1}m_\phi U$.

**Remark 1.** We obtain that let $U: X\to L^p(\Omega, \Theta, \mu_\Omega)$ be a structure-preserving homomorphism then the functional calculus and be defined by $\Phi(\psi)=U^{-1}m_{\psi\circ\phi}U$ and the reverse is also true, the functional calculus $\Phi$ defines a multiplication representation by $m_{\Upsilon(\psi)}=U\Phi(\psi)U^{-1}$, where *-homomorphism $\Upsilon: C(\Lambda)\to L^\infty(\Omega, \Theta, \mu_\Omega)$ defined by $\Upsilon(\psi)=\psi$ on $\Lambda$.

## REFERENCES

1. Ananova A. and Cont R., Pathwise integration with respect to paths of finite quadratic variation. Journal de Mathématiques Pures et Appliquées, 107(6): 737-757, (2017).
2. Arendt W., Vogt H., and Voigt J., Form Methods for Evolution Equations. Lecture Notes of the 18th International Internet seminar, version: 6 March (2019).
3. Budde C. and Landsman K., A bounded transform approach to self-adjoint operators: functional calculus and affiliated von Neumann algebras. Ann. Funct. Anal. 7, 3 (2016), 411–420.
4. Batty C., Gomilko A., and Tomilov Y., Product formulas in functional calculi for sectorial operators. Math. Z. 279, 1-2 (2015), 479–507.
5. Colombo F., Gentili G., Sabadini I., Struppa D.C., Noncommutative functional calculus: unbounded operators, preprint, (2007).
6. DeLaubenfels R., Automatic extensions of functional calculi. Studia Math. 114, 3 (1995), 237–259.
7. Dungey N., Asymptotic type for sectorial operators and an integral of fractional powers. J. Funct. Anal. 256, 5 (2009), 1387–1407.
8. Eisner T., Farkas B., Haase M., and Nagel R., Operator Theoretic Aspects of Ergodic Theory. Vol. 272 of Graduate Texts in Mathematics. Springer, Cham, (2015).
9. Friz P.K. and Zhang H., Differential equations are driven by rough paths with jumps, Journal of Differential Equations, 264:6226-6301, (2018).
10. Haase M., Functional analysis. An Elementary Introduction. Vol. 156 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, (2014).
11. Reed M. and Simon B., Methods of Modern Mathematical Physics I. Functional analysis. Second edition. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, (1980).
12. Ringrose J., On well-bounded operators. Journal of the Australian Mathematical Society, 1(3), (1960),

334-343. doi:10.1017/S1446788700026008.

13. Smart D. R., Eigenfunction expansions in Lp and C, Illinois Journal of Mathematics 3 (1959) 82-97.
14. Schmudgen K., Unbounded Self-adjoint Operators on Hilbert Space. Vol. 265 of Graduate Texts in Mathematics. Springer, Dordrecht, (2012).
15. Spain P.G., On well-bounded operators of type (B), Proc. Edinburgh Math. Soc. (2) 18 (1972), 35–48. MR 47:5648.
16. Takhtajan L., Quantum mechanics for mathematicians, A.M.S Grad. Studies in Math. 95, (2008).
17. Yaremenko M.I., Calderon-Zygmund Operators and Singular Integrals, Applied Mathematics & Information Sciences: Vol. 15: Iss. 1, Article 13, (2021).
18. Kowalski E., Lecture notes, online at https://www.math.ethz.ch/education/bachelor/lectures/fs2009/math/hilbert/.
19. Doust I. and Laubenfels R., Functional Calculus, Integral Representations, and Banach Space Geometry, Quaestiones Mathematicae, 17:2, 161-171, (1994). DOI: 10.1080/16073606.1994.9631755.