AJMS
Asian Journal of Mathematical Sciences

RESEARCH ARTICLE

AN IMPROVED MULTI-DERIVATIVE HYBRID LINEAR MULTISTEP METHODS FOR DIRECT SOLUTION OF THIRD-ORDER ORDINARY DIFFERENTIAL EQUATIONS

[*1]T.V. Abejide, [2]E.A. Areo, [3]A.L. Momoh

*[*1]Department of Basic Sciences, Federal Polytechnic Auchi, Edo State, Nigeria*
[2,3]Department of Mathematical Sciences, Federal University of Technology, Akure

Corresponding Email: tvabejide@auchipoly.edu.ng

Received: 19-05-2024; Revised: 24-06-2024; Accepted: 17-07-2024

ABSTRACT

This article considered an improved multi-derivative Hybrid linear multistep methods (IMDHLMM) for direct solution of third-order ordinary differential equations. Power series was used as the basis function in the derivation of the method. An approximate solution from the basis function was interpolated at some selected off-grid points while the third derivative of the approximate solution was collocated at all grid and off-grid points to generate a system of linear equations for the determination of the unknown parameters. The basic properties of the method such as order, consistency, zero stability, region of absolute stability and convergence was tested. The method was implemented in block mode to solve third order ordinary differential equations inlcuding Genesio equation to demonstrate the usability and efficiency of the methods. The absolute error obtained in the numerical experiments showed a better performance of the present method over some of the existing methods in the literature.

Keywords: Ordinary differential equation, third-order, hybrid, interpolation and collocation, boundary layer, genesio equation

INTRODUCTION

Over the years, attempts have been made at solving ordinary differential equations which frequently occur in different area of study and discipline with several authors coming out to develop their own methods of solution. Some of the solutions obtained by the authors are analytical, while others seeks to obtaining the numerical solutions to the the problems.

Hence, this article discussed the approximate solution to general third-order ordinary differential equations using the an Improved Multi-Derivative Hybrid Linear Multistep Methods (IMDHLMM). Thus, we seek to find numerical solution to equation of the form:

$$y'''(x) = f(x,\ y(x),\ y'(x),\ y''(x)), \qquad y(a) = y_0,\ y'(a) = \delta_0,\ y''(a) = y_N \tag{1}$$

where $a,\ y_0,\ \delta_0,\ y_N$, and f is a continuous function and satisfies a Lipschitz condition as given in [4].

The solution of (1) is conventionally solved by first reducing it to equivalent first-order differential equation, which have been considered by authors [1], [2], [3], and others. In like manner, authors such as [4], [5], [6], [7], [8], [9], to mention but few, have developed different method to solving equation (1) with each coming out with various degree of results.

In recent times, authors ([6], [7], [8], [9], [10]) have made tremendous progress in developing methods for the solution of (1) without reduction to equivalent first-order ordinary differential equation with each coming out with varying degree of successes in their approach. Many other numerical approaches has been considered by ([11], [12], [13], [14], [15], [16], [17], [18], [19])

While the search for more accurate numerical methods continues. This article develops an improved multi-derivative hybrid linear multistep methods (IMDHLMMs), implemented in block mode, for directly solution of equation (1). The article is arranged as follows: Section 1 is an introduction, Section 2 discusses how the methods were derived, and Section 3 analyses the method's basic properties. Section 4 gives numerical results to show the advantages of speed and accuracy. Section 5 provides a summary and conclusion.

Derivation of the Block Method

The solution of (1) is considered in the interval [0,4] by allowing $y(x)$ to be approximated by partial sum of power series polynomial of the form

$$y(x) \approx p(x) = \sum_{j=0}^{17} a_j x^j \tag{2}$$

where x is continuously differentiable and a_j's are parameters to be determine. In order to apply the procedure of collocation and interpolation to (2),third, fourth and fifth derivatives are obtained as:

$$y'''(x) \approx p'''(x) = \sum_{j=3}^{17} j(j-1)(j-2)a_j x^{j-3}, \tag{3}$$

$$y^{(4)}(x) = \sum_{j=4}^{17} j(j-1)(j-2)(j-3)a_j x^{j-4}, \tag{4}$$

and

$$y^{(5)}(x) = \sum_{j=5}^{17} j(j-1)(j-2)(j-3)(j-4)a_j x^{j-5}. \tag{5}$$

respectively. Note that the following are equivalent:

$$y'''(x,y(x),y'(x),y''(x)) \approx f(x,y(x),y'(x),y''(x)), y^{(4)}(x,y(x),y'(x),y''(x)) \approx g(x,y(x),y'(x),y''(x)),$$

,
and $y^{(5)}(x,y(x),y'(x),y''(x)) \approx \gamma(x,y(x),y'(x),y''(x))$.

Interpolating (2) at $x = x_{n+j}, j = 0,\ 1,\ \frac{7}{2}$ and collocating (3) to (5) at $x = x_{n+j}, j = 0,\ 1,\ 2{,}3,$ and 4 yield the system of equations that can be written in matrix form

$$X\,A = B \tag{6}$$

where,

$$X = \begin{pmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & \dots & x_n^{17} \\
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & \dots & x_{n+1}^{17} \\
1 & x_{n+\frac{7}{2}} & x^2_{n+\frac{7}{2}} & x^3_{n+\frac{7}{2}} & x^4_{n+\frac{7}{2}} & x^5_{n+\frac{7}{2}} & x^6_{n+\frac{5}{2}} & x^7_{n+\frac{7}{2}} & \dots & x^{17}_{n+\frac{7}{2}} \\
0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 & \dots & 4080x_n^{14} \\
0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 & \dots & 4080x_{n+1}^{14} \\
0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^4 & \dots & 4080x_{n+2}^{14} \\
0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & 210x_{n+3}^4 & \dots & 4080x_{n+3}^{14} \\
0 & 0 & 0 & 6 & 24x_{n+4} & 60x_{n+4}^2 & 120x_{n+4}^3 & 210x_{n+4}^4 & \dots & 4080x_{n+4}^{14} \\
0 & 0 & 0 & 0 & 24 & 60x_n & 120x_n^2 & 360x_n^3 & \dots & 57120x_n^{13} \\
0 & 0 & 0 & 0 & 24 & 60x_{n+1} & 120x_{n+1}^2 & 360x_{n+1}^3 & \dots & 57120x_{n+1}^{12} \\
0 & 0 & 0 & 0 & 24 & 60x_{n+2} & 120x_{n+2}^2 & 360x_{n+2}^3 & \dots & 57120x_{n+2}^{13} \\
0 & 0 & 0 & 0 & 24 & 60x_{n+3} & 120x_{n+3}^2 & 360x_{n+3}^3 & \dots & 57120x_{n+3}^{13} \\
0 & 0 & 0 & 0 & 24 & 60x_{n+4} & 120x_{n+4}^2 & 360x_{n+4}^3 & \dots & 57120x_{n+4}^{13} \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_n & 2520x_n^2 & \dots & 742560x_n^{12} \\
0 & 0 & 0 & 0 & 0 & 120 & 120x_{n+1} & 720x_{n+1}^2 & \dots & 742560x_{n+1}^{12} \\
0 & 0 & 0 & 0 & 0 & 120 & 120x_{n+2} & 720x_{n+2}^2 & \dots & 742560x_{n+2}^{12} \\
0 & 0 & 0 & 0 & 0 & 120 & 120x_{n+3} & 720x_{n+3}^2 & \dots & 742560x_{n+3}^{12} \\
0 & 0 & 0 & 0 & 0 & 120 & 120x_{n+4} & 720x_{n+4}^2 & \dots & 742560x_{n+4}^{12}
\end{pmatrix},$$

$$B = \left(y_n, y_{n+1}, y_{n+\frac{7}{2}}, f_n, f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}, g_n, g_{n+1}, g_{n+2}, g_{n+3}, g_{n+4}, \gamma_n, \gamma_{n+1}, \gamma_{n+2}, \gamma_{n+3}, \gamma_{n+4}\right)$$

$A = (a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, \dots\ a_{17})$. Solving equation (6) for a_j 's using Gaussian elimination methods and substitute into equation (2), yields after simple simplification the continuous scheme

$$y(t) = \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_{\frac{7}{2}} y_{n+\frac{7}{2}} + h^3 \sum_{j=0}^{3} \beta_j f_{n+j} + h^4 \sum_{j=0}^{3} \mu_j g_{n+j} + h^5 \sum_{j=0}^{3} \nu_j \gamma_{n+j} \tag{7}$$

where α_j, β_j, μ_j, and ν_j are the coefficients that define the scheme. The coefficients are presented in the appendix. Evaluating (7) and its first and second derivatives at $t = 1,\ 2,\ 3,\ 4$ yields the proposed improved multi-derivative hybrid Linear multistep method whose formulas are as presented in equations (8), (9), (10), and (11).

$$\left.\begin{array}{l}
y_3+\frac{2}{7}\,y_0-\frac{3}{5}\,y_1-\frac{24}{35}\,y_{\frac{7}{2}}=*\,2mm\\
+h^5\left(-\frac{1409710531}{850195906560}\,\gamma_3-\frac{3416922763}{503819796480}\,\gamma_2-\frac{3915971533}{108825076039680}\,\gamma_0+\frac{631116727}{108825076039680}\,\gamma_4-\frac{1669553929}{425097953280}\,\gamma_1\right)\\
+h^4\left(-\frac{5549931977}{72550050693120}\,g_4+\frac{5771699359}{425097953280}\,g_3-\frac{12002518379}{850195906560}\,g_1-\frac{4006875659}{6218575773696}\,g_0-\frac{1751089}{1761607680}\,g_2\right)\\
+h^3\left(-\frac{20462244857}{77510737920}\,f_2-\frac{40687537171}{463743221760}\,f_3-\frac{2216650977353}{652950456238080}\,f_0+\frac{179968881137}{652950456238080}\,f_4-\frac{9616126523}{66249031680}\,f_1\right)
\end{array}\right\}$$

$$\left.\begin{array}{l}
y_4-\frac{3}{7}\,y_0+\frac{4}{5}\,y_1-\frac{48}{35}\,y_{\frac{7}{2}}=*\,2mm\\
+h^5\left(\frac{5022074429}{425097953280}\,\gamma_3+\frac{1026254197}{251909898240}\,\gamma_2+\frac{2098758707}{54412538019840}\,\gamma_0+\frac{6645846967}{54412538019840}\,\gamma_4+\frac{1546338551}{212548976640}\,\gamma_1\right)\\
+h^4\left(-\frac{48035230175}{21765015207936}\,g_4-\frac{923402071}{30364139520}\,g_3+\frac{1781215619}{60728279040}\,g_1+\frac{9253967431}{12091675115520}\,g_0-\frac{1751089}{880803840}\,g_2\right)\\
+h^3\left(\frac{15056727047}{38755368960}\,f_2+\frac{11763238763}{33124515840}\,f_3+\frac{1426133897143}{326475228119040}\,f_0+\frac{3822753755633}{326475228119040}\,f_4+\frac{55717322851}{231871610880}\,f_1\right)
\end{array}\right\},$$

and additional schemes.

$$\left.\begin{array}{l}
hy'_0+\frac{9}{7}\,y_0+\frac{4}{35}\,y_{\frac{7}{2}}-\frac{7}{5}\,y_1=*\,2m\\
+h^5\left(\frac{825232412987}{1585736822292480}\,\gamma_0-\frac{83472199649}{1585736822292480}\,\gamma_4+\frac{162487855967}{6194284462080}\,\gamma_1-\frac{158550905299}{7341374177280}\,\gamma_2+\frac{60478318517}{12388568924160}\,\gamma_3\right)\\
+h^4\left(\frac{417824978509}{12388568924160}\,g_1+\frac{2873324514901}{288315785871360}\,g_0+\frac{4655414249}{215922769920}\,g_2-\frac{3277715471}{112623353856}\,g_3+\frac{726688348831}{1057157881528320}\,g_4\right)\\
+h^3\left(\frac{37934112731417}{74331413544960}\,f_1+\frac{570462138117151}{9514420933754880}\,f_0-\frac{23527134155959}{9514420933754880}\,f_4-\frac{1063525082333}{14682748354560}\,f_2+\frac{6537044514007}{74331413544960}\,f_3\right)\\
\\
hy'_1+\frac{5}{7}\,y_0-\frac{3}{5}\,y_1-\frac{4}{35}\,y_{\frac{7}{2}}=\\
+h^5\left(-\frac{19312310167}{17343996493824}\,\gamma_3+\frac{7767827905}{10277923848192}\,\gamma_2-\frac{206716531129}{2220031551209472}\,\gamma_0+\frac{22726604347}{2220031551209472}\,\gamma_4-\frac{5588703577}{667076788224}\,\gamma_1\right)\\
+h^4\left(-\frac{593423765455}{4440063102418944}\,g_4+\frac{63570336899}{8671998246912}\,g_3-\frac{302747477407}{17343996493824}\,g_1-\frac{818247378805}{493340344713216}\,g_0+\frac{1558624493}{302291877888}\,g_2\right)\\
+h^3\left(-\frac{2079355099441}{20555847696384}\,f_2-\frac{276512096935}{9460361723904}\,f_3-\frac{115660184370629}{13320189307256832}\,f_0+\frac{70330226423}{146375706673152}\,f_4-\frac{28938028182307}{104063978962944}\,f_1\right)\\
\\
hy'_2+\frac{1}{7}\,y_0+\frac{1}{5}\,y_1-\frac{12}{35}\,y_{\frac{7}{2}}=\\
+h^5\left(-\frac{90290160089}{86719982469120}\,\gamma_3-\frac{882540593}{133479530496}\,\gamma_2-\frac{192671492567}{11100157756047360}\,\gamma_0+\frac{39230008693}{11100157756047360}\,\gamma_4-\frac{94344711883}{43359991234560}\,\gamma_1\right)\\
+h^4\left(-\frac{1055359397153}{22200315512094720}\,g_4+\frac{43879985029}{4817776803840}\,g_3-\frac{30120549379}{3211851202560}\,g_1-\frac{992293320949}{3171473644584960}\,g_0-\frac{1751089}{3523215360}\,g_2\right)\\
+h^3\left(-\frac{522975129497}{1868713426944}\,f_2-\frac{5547436185479}{104063978962944}\,f_3-\frac{22115704584527}{13320189307256832}\,f_0+\frac{2329817972071}{13320189307256832}\,f_4-\frac{8534800286057}{104063978962944}\,f_1\right)\\
\\
hy'_3-\frac{3}{7}\,y_0+y_1-\frac{4}{7}\,y_{\frac{7}{2}}=\\
+h^5\left(\frac{7739157091}{7883634769920}\,\gamma_3+\frac{410760458201}{51389619240960}\,\gamma_2+\frac{588049769711}{11100157756047360}\,\gamma_0-\frac{95362905149}{11100157756047360}\,\gamma_4+\frac{259151057411}{43359991234560}\,\gamma_1\right)\\
+h^4\left(\frac{874074700867}{7400105170698240}\,g_4-\frac{103717630321}{8671998246912}\,g_3+\frac{24264676829}{1126233538560}\,g_1+\frac{1627938575639}{1707716577853440}\,g_0-\frac{9295556827}{1511459389440}\,g_2\right)\\
+h^3\left(\frac{34312142375479}{102779238481920}\,f_2+\frac{13907713386139}{520319894814720}\,f_3+\frac{335944191899587}{66600946536284160}\,f_0-\frac{29901757410043}{66600946536284160}\,f_4+\frac{1474234389097}{6757401231360}\,f_1\right)\\
\\
hy'_4-y_0+\frac{9}{5}\,y_1-\frac{4}{5}\,y_{\frac{7}{2}}=\\
+h^5\left(\frac{345432282353}{7883634769920}\,\gamma_3-\frac{366013699741}{51389619240960}\,\gamma_2+\frac{596429599669}{11100157756047360}\,\gamma_0+\frac{7421164890641}{11100157756047360}\,\gamma_4+\frac{25021068923}{1238856892416}\,\gamma_1\right)\\
+h^4\left(-\frac{281642390890829}{22200315512094720}\,g_4-\frac{363781233061}{3941817384960}\,g_3+\frac{7555422985027}{86719982469120}\,g_1+\frac{154143505981}{117461986836480}\,g_0-\frac{6818066821}{302291877888}\,g_2\right)\\
+h^3\left(\frac{81973160169037}{102779238481920}\,f_2+\frac{567797677212377}{520319894814720}\,f_3+\frac{43494273188117}{5123149733560320}\,f_0+\frac{4967805682923271}{66600946536284160}\,f_4+\frac{318144558684727}{520319894814720}\,f_1\right)
\end{array}\right\}$$

$$
\left.
\begin{aligned}
&h^2y''_0-\frac{4}{7}y_0+\frac{4}{5}y_1-\frac{8}{35}y_{\frac{7}{2}}=\\
&+h^5\left(-\frac{228160754671}{108825076039680}\gamma_0+\frac{58863752119}{326475228119040}\gamma_4-\frac{101782089481}{1275293859840}\gamma_1+\frac{119286064501}{1511459389440}\gamma_2-\frac{4638385771}{283398635520}\gamma_3\right)\\
&+h^4\left(-\frac{290221147499}{2550587719680}g_1-\frac{28979842185089}{652950456238080}g_0-\frac{65243095903}{755729694720}g_2+\frac{122663007391}{1275293859840}g_3-\frac{512238861769}{217650152079360}g_4\right)\\
&+h^3\left(-\frac{578340594101}{437243609088}f_1-\frac{32340464715877}{93278636605440}f_0+\frac{2368386895943}{279835909816320}f_4+\frac{1620877881}{3690987520}f_2-\frac{202737902797}{728739348480}f_3\right)\\[2ex]
&h^2y''_1-\frac{4}{7}y_0+\frac{4}{5}y_1-\frac{8}{35}y_{\frac{7}{2}}=\\
&+h^5\left(\frac{2671945277}{2550587719680}\gamma_3-\frac{5158143311}{302291877888}\gamma_2+\frac{30277403699}{326475228119040}\gamma_0-\frac{193213291}{12091675115520}\gamma_4+\frac{803265277}{425097953280}\gamma_1\right)\\
&+h^4\left(\frac{27333529121}{130590091247616}g_4-\frac{5557193569}{1275293859840}g_3-\frac{115763922283}{2550587719680}g_1+\frac{68884758673}{43530030415872}g_0+\frac{26987977889}{755729694720}g_2\right)\\
&+h^3\left(-\frac{27840131411}{77510737920}f_2-\frac{21757699933}{3060705263616}f_3+\frac{2218942965457}{279835909816320}f_0-\frac{493391469061}{652950456238080}f_4-\frac{718691098261}{5101175439360}f_1\right)\\[2ex]
&h^2y''_2-\frac{4}{7}y_0+\frac{4}{5}y_1-\frac{8}{35}y_{\frac{7}{2}}=\\
&+h^5\left(-\frac{433603879}{510117543936}\gamma_3+\frac{607714573}{167939932160}\gamma_2+\frac{20177585203}{326475228119040}\gamma_0+\frac{54225541}{29679566192640}\gamma_4+\frac{11573170423}{1275293859840}\gamma_1\right)\\
&+h^4\left(-\frac{3585110495}{130590091247616}g_4+\frac{651833969}{85019590656}g_3+\frac{31297737607}{850195906560}g_1+\frac{68417183197}{59359132385280}g_0-\frac{54446957407}{755729694720}g_2\right)\\
&+h^3\left(\frac{16858566317}{77510737920}f_2-\frac{619289560273}{15303526318080}f_3+\frac{2463681702283}{391770273742848}f_0+\frac{216048890353}{1958851368714240}f_4+\frac{4844614083649}{15303526318080}f_1\right)\\[2ex]
&h^2y''_3-\frac{4}{7}y_0+\frac{4}{5}y_1-\frac{8}{35}y_{\frac{7}{2}}=\\
&+h^5\left(\frac{1795414421}{283398635520}\gamma_3+\frac{36729578869}{1511459389440}\gamma_2+\frac{8663608337}{108825076039680}\gamma_0-\frac{146205193}{5022695817216}\gamma_4+\frac{9153188087}{1275293859840}\gamma_1\right)\\
&+h^4\left(\frac{87585604151}{217650152079360}g_4-\frac{95051058017}{1275293859840}g_3+\frac{5747618783}{231871610880}g_1+\frac{907182213247}{652950456238080}g_0+\frac{26987977889}{755729694720}g_2\right)\\
&+h^3\left(\frac{4103817603}{5167382528}f_2+\frac{23375083583}{56056872960}f_3+\frac{4671543936317}{652950456238080}f_0-\frac{2998143356431}{1958851368714240}f_4+\frac{4334113023041}{15303526318080}f_1\right)\\[2ex]
&h^2y''_4-\frac{4}{7}y_0+\frac{4}{5}y_1-\frac{8}{35}y_{\frac{7}{2}}=\\
&+h^5\left(\frac{2916136369}{33124515840}\gamma_3-\frac{15478171741}{215922769920}\gamma_2-\frac{7617937193}{65295045623808}\gamma_0+\frac{26120604821}{12091675115520}\gamma_4+\frac{2090793113}{85019590656}\gamma_1\right)\\
&+h^4\left(-\frac{29750356752731}{652950456238080}g_4-\frac{2804302625}{19619905536}g_3+\frac{319664208533}{2550587719680}g_1-\frac{7297162073}{6218575773696}g_0-\frac{65243095903}{755729694720}g_2\right)\\
&+h^3\left(-\frac{321302867}{77510737920}f_2+\frac{24467245316911}{15303526318080}f_3-\frac{4044250869833}{1958851368714240}f_0+\frac{46112281095679}{130590091247616}f_4+\frac{2827606827371}{5101175439360}f_1\right)
\end{aligned}
\right\}
$$

Analysis of The Method

Here, we examine the analysis of the basic properties of the proposed method, including order, error constant, consistency, zero stability, and convergence.

Local Truncation Error And Order

In line with what has established in author [4], let the linear difference operator L associated with the IMDHLMM be defined as

$$L[y(x_n);h] = \sum_{j=0}^{k} \left(\alpha_j y(x_n + jh) - h^3 \beta_j y'''(x_n + jh) - h^4 \gamma_j y^{iv}(x_n + jh) - h^5 \mu_j y^{v}(x_n + jh)\right) \tag{12}$$

where $y(x)$ is an arbitrary test function that is continuously differentiable in the interval [a,b], α_j, β_j, γ_j and μ_j are the continuous coefficients. Expanding $(x_n + jh)$, $y'''(x_n + jh)$, $y^{iv}(x_n + jh)$ and $y^{v}(x_n + jh)$ for $j = 0,\ 1, \dots, m$ in Taylor's series about x_n and collecting the like terms in h and y gives

$$L[y(x);h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + C_3 h^3 y'''(x) + \cdots + C_p h^p y^{(p)}(x) + \cdots \tag{13}$$

The difference operator L and the associated multi-derivative linear multistep methods are said to be of order p if $C_0 = C_1 = C_2 = \cdots = C_p = C_{p+1} = C_{p+2} = 0$ and $C_{p+3} \neq 0$ while the term C_{p+3} is called the **error constant** and the **local truncation** is given by $\tau_k = C_{P+3} h^{(p+3)} y^{(p+3)}(x_n) + O(h^{(p+4)}$ Hence, the order and error constants associated with the methods developed are given below.

Table 1: Order and error IMDHLMM

Scheme	Order	Error Constant
(8)	15	$-\frac{35926318}{81318721447280}$
(9)	15	$-\frac{9600089}{14397557560049}$
(10)	15	$\frac{1369427567}{3428616727687908}$
	15	$-\frac{32898343}{42000554914176}$
	15	$-\frac{9600089}{57590230240198}$
	15	$\frac{1889565353}{4200055491417687}$
	15	$-\frac{18175756}{42000554914176}$
(11)	15	$-\frac{207177413}{15000198183634}$
	15	

	$\frac{1176863566}{1050013872854421}$
15	$\frac{1803293763}{1050013872854421}$
15	$\frac{1176863566}{1050013872854421}$
15	$-\frac{207177413}{15000198183634}$

3.2 Consistency

IMDHLMM is said to be consistent if the following conditions are satisfied according to [17] :

1. the order $\rho \geq 1$
2. $\sum_{j=0}^{k} \alpha_j = 0$
3. $\rho(1) = \rho'(1) = 0$
4. $\rho'''(1) = 3!\,\sigma(1)$

The consistency of IMDHLMM examine as follows;

(i) the order of IMDHLMM is $p = 15$

(ii) The α's are; $\alpha_0 = -\frac{3}{7},\ \alpha_1 = \frac{4}{5},\ \alpha_{\frac{7}{2}} = -\frac{48}{35},\ \alpha_4 = 1,$

$$\sum_{j=0}^{k} \alpha_j = \alpha_0 + \alpha_1 + \alpha_{\frac{7}{2}} + \alpha_4 = -\frac{3}{7} + \frac{4}{5} - \frac{48}{35} + 1 = 0$$

(iii) Also, $\rho(r)$ is the first characteristic polynomial here

$\rho(r) = r^4 + \frac{4r}{5} - \frac{48r^{\frac{7}{2}}}{35} - \frac{3}{7} = 0$, when $r = 1$

$\rho'(1) = 4r^3 + \frac{4}{5} - \frac{24r^{\frac{5}{2}}}{5} = 0$, when $r = 1$

(iv) Again, $\rho'''(1) = 24r - 18r^{\frac{1}{2}} = 6,$

$\sigma(r) = \frac{3822753755633}{326475228119040} r^4 + \frac{11763238763}{33124515840} r^3 + \frac{15056727047}{38755368960} r^2 + \frac{55717322851}{231871610880} r + \frac{1426133897143}{326475228119040}$, when

$r = 1$, $3!\,\sigma(1) = 6$, hence, $\rho'''(1) = 6!\,\sigma(1)$

Thus, conditions (1) - (4) are satisfied. This implies that IMDHLMM is consistent.

3.3 Zero Stability

Definition 3.3.1: The linear multistep method is said to be zero stable if no root of the first characteristic polynomial $\rho(r)$ has a modulus greater than one and if every root of modulus one has multiplicity not more significant than three (see [18]).

$$\overline{\rho}(r) = det[r\bar{A} - \bar{E}] \qquad (14)$$

satisfies $|r_s| \leq 1$ and every root with $|r_s| = 1$ has multiplicity not exceeding three in the limit as $h \to 0$. The first characteristic polynomial for equation (8) is given by:

$$\rho(r) = r^4 + \frac{4r}{5} - \frac{48r^{\frac{7}{2}}}{35} - \frac{3}{7} \qquad (15)$$

equating equation (15) to zero and solving for r gives

$$r^4 + \frac{4r}{5} - \frac{48r^{\frac{7}{2}}}{35} - \frac{3}{7} = 0$$

$$r = (1,1,1)$$

The root r of equation (15) for which $|r| = 1$ is simple (since the multiplicity of the root r is three), hence the method for is zero stable as $h \to 0$ by definition (see [17]).

3.4 Region Of Absolute Stability

The region of absolute stability of IMDHLMM

$$\left.\begin{aligned}
&y_4 - \frac{3}{7}\, y_0 + \frac{4}{5}\, y_1 - \frac{48}{35}\, y_{\frac{7}{2}} =* \\
&+h^5\left(\frac{5022074429}{425097953280}\,\gamma_3 + \frac{1026254197}{251909898240}\,\gamma_2 + \frac{2098758707}{54412538019840}\,\gamma_0 + \frac{6645846967}{54412538019840}\,\gamma_4 + \frac{1546338551}{212548976640}\,\gamma_1\right) \\
&+h^4\left(-\frac{48035230175}{21765015207936}\,g_4 - \frac{923402071}{30364139520}\,g_3 + \frac{1781215619}{60728279040}\,g_1 + \frac{9253967431}{12091675115520}\,g_0 - \frac{1751089}{880803840}\,g_2\right) \\
&+h^3\left(\frac{15056727047}{38755368960}\,f_2 + \frac{11763238763}{33124515840}\,f_3 + \frac{1426133897143}{326475228119040}\,f_0 + \frac{3822753755633}{326475228119040}\,f_4 + \frac{55717322851}{231871610880}\,f_1\right)
\end{aligned}\right\}$$

is obtained by first considering the following two characteristic polynomials:

$$P(r) = r^4 + \frac{4r}{5} - \frac{48r^{\frac{7}{2}}}{35} - \frac{3}{7} \qquad (16)$$

and

$$Q(r) = \frac{314210038481}{32647522811904}r^4 + \frac{858336057361}{2550587719680}r^3 + \frac{17889848827}{45801799680}r^2 + \frac{100893952853}{364369674240}r + \frac{844291785011}{163237614059520} \qquad (17)$$

The stability polynomial of IMDHLMM is given by

$$\Pi(r,z) = P(r) + zQ(r) \tag{18}$$

$$= r^4 + \frac{4r}{5} - \frac{48r^{\frac{7}{2}}}{35} - \frac{3}{7} - z\left(\frac{314210038481}{32647522811904}r^4 + \frac{858336057361}{2550587719680}r^3 + \frac{17889848827}{45801799680}r^2 + \frac{100893952853}{364369674240}r + \frac{844291785011}{163237614059520}\right)$$

Using $r = e^{I\theta}$ in (18) and setting to zero yields the following expression for z after simplification.

$$z = -\frac{4663931830272\left(48\left(e^{I\theta}\right)^{\frac{7}{2}} - 35\left(e^{I\theta}\right)^4 - 28\,e^{I\theta} + 15\right)}{844291785011 + 1571050192405(e^{I\theta})^4 + 54933507671104(e^{I\theta})^3 + 63759421219428(e^{I\theta})^2 + 45200490878144\,e^{I\theta}} \tag{19}$$

(19) is then plotted using the following MapleSoft codes

$complexplot(z, \theta = 0..2\pi, filled = true, labels = [``\text{Re}", ``\text{Im}"], color = grey)$

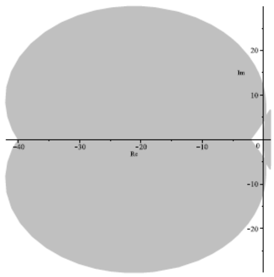


Figure 1: Region of absolute stability of IMDHLMM

Numerical Examples

In this section, we test some linear and nonlinear numerical examples to illustrate the accuracy of the methods. The maximum absolute error is computed as maxERR= $Max|y(xi) - y_i|, i = 1, \ldots, N,$ where $y(x_i)$ is the exact solution computed at the grid point and y_i is an approximation to the exact solution using the IMDHLMM. For each example, we find the absolute errors of the approximate solutions and were compared with various existing methods in the literature. The accuracy of our method is seen in the small error values obtained.

: The first test example considered is the linear third-order ODE

$$y''' = -y, \quad y(0) = 1, y'(0) = -1, y''(0) = 1, \; 0 \le x \le 1, \; h = 0.1$$

whose exact solution is $y(x) = e^{-x}$. The numerical solution was obtained in the interval [0,1] over ten iterations. The absolute errors of (IMDHLMM) are presented in Table 2 and Figure 2 as compared with those of [18].

Table 2: **Comparison of absolute error of problem 1 using $h = 0.1$**

N	maxErr (IMDHLMM)	Error in [18]
0.1	$1.0E-27$	$2.8160E-24$
0.2	$1.0E-27$	$1.1025E-23$
0.3	$2.0E-27$	$2.4162E-23$
0.4	$2.0E-27$	$1.797E-23$
0.5	$1.0E-27$	$6.3522E-23$
0.6	$3.0E-27$	$8.8946E-23$
0.7	$3.0E-27$	$1.1768E-22$
0.8	$2.0E-27$	$1.4936E-22$
0.9	$2.0E-27$	$1.8358E-22$
1	$3.0E-27$	$2.1997E-22$

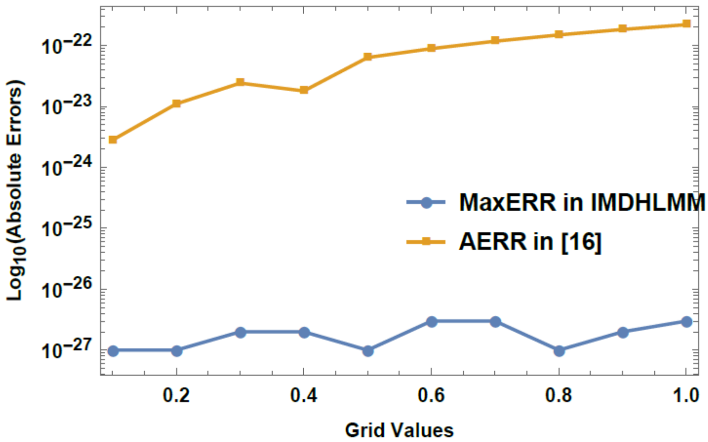


Figure 2: Graph of comparison of results in Table 2

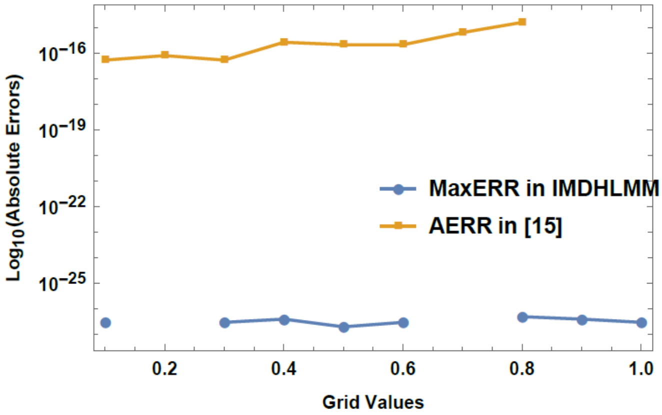


Figure 3: Graph of comparison of results in Table 3

Problem 2: The second test example considered is the oscillatory problem

$$y''' = 3sinx, \qquad y(0) = 1;\ \ y'(0) = 0;\ \ y''(0) = -2;\ \ h = 0.1$$

with the theoretical solution $y(x) = 3cosx + \frac{x^2}{2} - 2,$.

The proposed method (IMDHLMM) was applied to solve the second example in [0,1] over ten iterations and the absolute error maxERR are compared with those of [17] in the Table 3 and Figure 3.

Table 3: **Comparison of absolute error of problem 2 using $h = 0.1$**

N	maxErr (IMDHLMM)	Error in [17]
0.1	$3.0E-27$	$5.5511E-17$
0.2	0	$8.3266E-17$
0.3	$3.0E-27$	$5.5511E-17$
0.4	$4.0E-27$	$2.7755E-16$
0.5	$2.0E-27$	$2.2204E-16$
0.6	$3.0E-27$	$2.2204E-16$
0.7	0	$6.6613E-16$
0.8	$5.0E-27$	$1.6653E-15$
0.9	$4.0E-27$	
1	$3.0E-27$	

Problem 3: We look at another problem

$$y''' = e^x, \quad y(0) = 3, \qquad y'(0) = 1, \qquad y''(0) = 5, \qquad h = 0.1$$

$$Exact: y(x) = 2 + 2x^2 + e^x$$

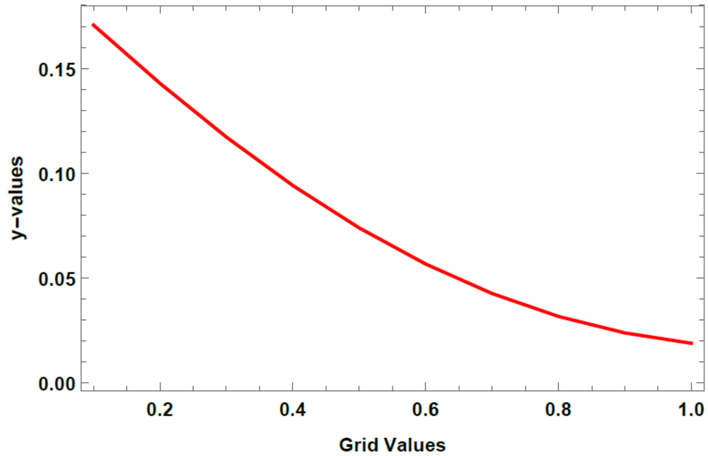
We compare the results obtained with that of Duromola 2022

Table 4: **showing the exact solution and computed results from the proposed method for the problem in example 2. With $h = 0.1$**

N	IMDHLMM computed	Exact
0.1	3.1251709180756476248117078 3	3.12517091807564762481170783
0.2	3.30140275816016983392107200	3.30140275816016983392107199
0.3	3.52985880757600310398374431	3.52985880757600310398374431
0.4	3.81182469764127031782485295	3.81182469764127031782485295
0.5	4.14872127070012814684865079	4.14872127070012814684865079
0.6	4.54211880039050897487536767	4.54211880039050897487536767
0.7	4.99375270747047652162454939	4.99375270747047652162454939
0.8	5.50554092849246760457953753	5.50554092849246760457953753
0.9	6.07960311115694966380012656	6.07960311115694966380012656
1	6.71828182845904523536028747	6.71828182845904523536028747

Table 5: **Comparing the error obtained from the proposed method with that of Duromola 2022 for the problem in example 3. With $h = 0.1$**

N	maxErr (IMDHLMM)	Error in Duromola (2022)
.1	0	$2.8160E-24$
.2	$1.10E-26$	$1.1025E-23$
.3	0	$2.4162E-23$
.4	0	$1.797E-23$
.5	$1.10E-26$	$6.3522E-23$
.6	0	$8.8946E-23$
.7	0	$1.1768E-22$
.8	0	$1.4936E-22$
.9	0	$1.8358E-22$
	0	$2.1997E-22$



Problem 4: Application to solve nonlinear Genesio equation as seen in [17]. The chaotic Genesio equation is given as:

$$y''' = -\alpha y'' - \beta y' + f(y(x))$$

where, $f(y(x)) = -\gamma y(x) + y^2(x)$ $y(0) = 0.2$, $y'(0) = -0.3$, $y'' = 0.1$, $x \in [a, b]$, $\alpha = 1.2$, $\beta = 2.92$ and $\gamma = 6$ are positive constants that satisfied $\alpha\beta < \gamma$. The solution of the Genesio eqaution is considered considered in [0,1] over ten iteration. The results are presented in Table 6 and further in Figure 4.

Table 6: **Solution of Genesio equation**

N	(IMDHLMM) Computed
0.1	0.1706088593
0.2	0.1428350336
0.3	0.1171963714
0.4	0.09410082038
0.5	0.07384747538
0.6	0.05663133027
0.7	0.04255034761
0.8	0.03161211329
0.9	0.02374189440
1.0	0.01879379378

Problem 5: Application to a boundary layer problem

$$2y''' + yy'' = 0, \qquad y(0) = y'(0) = 0;\ y''(0) = 1$$

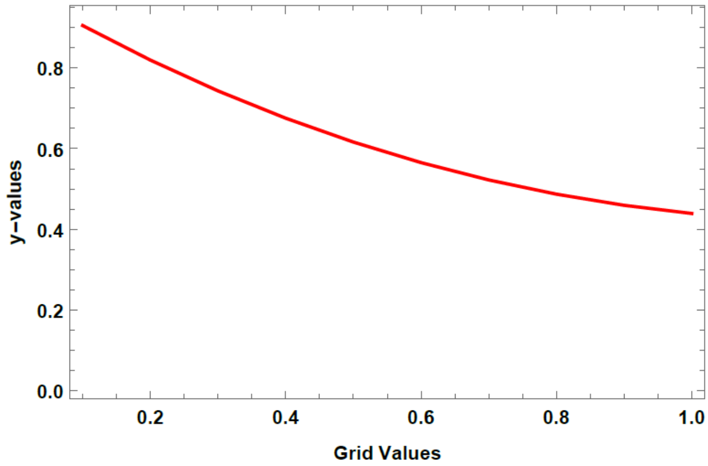


Figure 5: Graph of results (Boundary layer) displayed in Table 7

There is no known exact solution for the problem. The solution of the boundary layer problem is considered in [0,1] over ten iteration. The results are presented in Table 7 and further in Figure 5.

Table 7: **Solution of Boundary layer problem**

N	(IMDHLMM) Computed
0.1	0.904919642523
0.2	0.819379641120
0.3	0.742977018060
0.4	0.675361369909
0.5	0.616225463831
0.6	0.565299233729
0.7	0.522343555389
0.8	0.487144489263
0.9	0.459508405424
1.0	0.439258012024

CONCLUSION

This work has contributed an improved multi-derivative hybrid linear multistep method (IMDHLMM) to solve third-order ordinary differential equations directly. The method is zero and p-stable, consistent and convergent. The numerical results obtained from solved problems show the method's efficiency and accuracy advantages over existing methods in the literature. The results in Tables 2-7 and Figures 2-5 show that the method is better in accuracy and can compete well with others in the literature for solving similar problems.

REFERENCES

1. Kayode S.J, Duromola M. K and Bolarinwa Bolaji. (2014). Direct solution of initial value problems of fourth order ordinary differential equations using modified implicit hybrid block method. Journal of Scientific Research and Reports.3 (21); 2792-2800.

2. Agbeboh, G.U. and Omonkaro, B. (2009), On the solution of singular initial value problems in ordinary differential equations using a new third order inverse Runge-Kutta method; International Journal of Physical Sciences Vol. 5 (4), pp. 299-307, April 2010.
3. Olabode, B. T. (2013). Block multistep method for the direct solution of third order of ordinary differential equations. FUTA Journal of Research in sciences, 2(2013), 194-200.
4. Jator, S. N. (2007). A Sixth-Order Linear Multistep Method for third-order ordinary differential equations. International journal of pure and applied Mathematics, 40(4), 457-472.
5. Bolarinwa Bolaji (2015). Fully implicit Block-Predictor Corrector method for the Numerical Integration of $y''' = f(x, y, y', y'')y(a) = \psi 1, y'(a) = \psi 2, y''(a) = \psi 3$, Journal of Scientific Research and Reports 6(2):165-171.
6. Awoyemi, D.O. (2003), A P-stable linear multi-step method for solving general third order ordinary differential equations, Intern. J. Comput. Math, 80(8) 987-993
7. Akinfenwa, O.A. and Jator, S.N. (2012), continuous third derivative formulas for second order boundary value problems; Intern J. of computation and applied maths.Vol.7, No.3 pp. 371-390
8. Jator, S.N. (2008), Multiple finite difference methods for solving third order ordinary differential equations; International Journal of Pure and Applied Mathematics, 43, No. 2, 253-265
9. Kayode S.J, Duromola M. K and Bolarinwa Bolaji (2014). Direct solution of initial value problems of fourth order ordinary differential equations using modified implicit hybrid block method. Journal of Scientific Research and Reports.3 (21); 2792-2800.
10. Awoyemi, D., Kayode, S. and Adoghe, L. (2014). A five-step P-stable method for the numerical integration of third order ordinary differential equations. American J. Comput. Math. 2014, 4, 119-126.
11. Kayode, S.J. and Obarhua, F.O. (2017) Symmetric 2-Step 4-Point Hybrid Method for the Solution of General Third Order Differential Equations. Journal of Applied and Computational Mathematics, 6, 348
12. Allogmany, R. and Ismail, F. (2020): Implicit Three-Point Block Numerical Algorithm for Solving Third Order Initial Value Problem Directly with Applications. Mathematics, 8(10), p.1771.
13. Areo, E.A. and Abejide, T.V. (2018), Multi-derivative Linear Multi-step Methods for Solving Third Order Boundary Value Problems. Journal of Advances in Mathematics and Computer Science 28(3): x-xx, Article no.JAMCS.42045 ISSN: 2456-9968
14. Areo, E.A. and Adeniyi, R.B. (2013), Sixth-Order Hybrid Block Method for the Numerical Solution of First Order Initial Value Problems; Journal of Mathematical Theory and Modeling Vol.3, No.8
15. Badmus A.M. and Yahaya Y.A. (2014). New Algorithm of Obtaining Order and Error Constants of Third Order Linear Multistep Method. Asian Journal of Fuzzy and Applied Mathematics;2(6), ISSN: 2321-564X
16. Bhrawy, A.H. and Abd-Elhameed, W.M. (2011): New Algorithm for the Numerical Solutions of Nonlinear Third-Order Differential Equations Using Jacobi-Gauss Collocation Method; Hindawi Publishing Corporation Mathematical Problems in Engineering Volume, Article ID 837218, 14 pages Applied mathematics letters 15 445-451
17. Duromola, M. and Momoh, L. A. (2019): Hybrid Numerical Method with Block Extension for Direct Solution of Third Order Ordinary Differential Equations, American Journal of Applied Mathematics and Statistics; DOI: 10.4236/ajcm.2019.92006
18. Duromola, M. (2022): Single-Step Block Method of P-Stable for Solving Third-Order Differential Equations (IVPs): Ninth Order of Accuracy, American Journal of Applied Mathematics and Statistics; DOI: 10.12691/ajams-10-1-2
19. Jame, A.A. (2013), Half-Step continuous block method for the solutions of modeled problems of ordinary differential equations. American Journal of Computational Mathematics, 3, 261-269