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On the Application of the Fixed Point Theory to the Solution of Systems of Linear Differential Equations to Biological and Physical Problems

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ABSTRACT

In this study, I worked on how to solve the biological and physical problems using systems of linear differential equations. A differential equation is an equation involving an unknown function and one or more of its derivatives. In this work, we consider systems of differential equations and their underlying theories illustrated with some solved examples. Finally, two applications, one to cell biology and the other to physical problems, were considered precisely.

Key words: Differential equations, biological, linear system, eigenvalues, characteristic equations

INTRODUCTION

If each of the coefficients in a system of m linear equations in n unknowns is a linear differential operator defined on an interval I and if each of the quantities on the right-hand side of the equation is a continuous function on I we then have what is known as a system of m linear differential equations in n unknowns. Such a system has[7] the form.

$$\begin{matrix} L_{11}x_1+\ldots+ & L_{1n}x_n = & h_1(t) \\ \vdots & \vdots & \vdots \\ L_{m1}x_1+\ldots+ & L_{mn}x_n = & h_m(t) \end{matrix} \tag{1.1}$$

Where, the L_{ij} are linear differential operators defined on I and $x_1,\ldots x_n$ are unknown functions of t. As usual, we say that a system like this is homogeneous if all of the h_i are identically zero and non-homogeneous otherwise. Solutions of these systems are particularly important in application and as we shall see, arise in such diverse fields as biology, economics, and physics.

The formal similarity between (1.1) and an ordinary system of linear equations suggests that we[11] rewrite (1.1) in matrix form as.

$$LX = H(t) \tag{1.2}$$

Where, L is the $m\ x\ n$ operator matrix

$$\begin{bmatrix} L_{11} & \ldots & L_{tn} \\ \vdots & & \vdots \\ L_{m1} & \ldots & L_{mn} \end{bmatrix}$$

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and X and $H(t)$ are the column vectors

$$X=\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ and } H(t)=\begin{bmatrix} h_1(t) \\ \vdots \\ h_m(t) \end{bmatrix}$$

for then

$$\begin{bmatrix} L_{11} & \dots & L_{1n} \\ \vdots & & \vdots \\ L_{m1} & \dots & L_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}=\begin{bmatrix} L_{11}x_1 & +\dots+ & L_{1n}x_n \\ \vdots & & \vdots \\ L_{m1}x_1 & +\dots+ & L_{mn}x_n \end{bmatrix} \tag{1.3}$$

Moreover, (1.2) is an alternative version of (1.1). Of course, this kind of manipulation does not mean much until we have introduced suitable vector spaces and an appropriate linear transformation between them. However, this is easily done. We[1] just let r_m and r_n be the spaces of column vectors of the form.

$$H(t)=\begin{bmatrix} h_1\ (t) \\ \vdots \\ h_m\ (t) \end{bmatrix}, \quad X(t)\begin{bmatrix} x_1(t) \\ \vdots \\ h_m\ (t) \end{bmatrix},$$

Respectively, where, the $h_i\,(t)$ is continuous on I and the $x_i\,(t)$ sufficiently differentiable on I so that the L_{ij} can be applied to them, and where addition and scalar multiplication are defined component wise as usual. This done, we let L: $r_n \rightarrow r_m$ be the linear transformation defined by (1.3); that is.

$$L(x)=\begin{bmatrix} L_{11}x_1 & +\dots+ & L_{1n}x_n \\ \vdots & & \vdots \\ L_{m1}x_1 & +\dots+ & L_{mn}x_n \end{bmatrix}$$

Then, the original system can indeed be written in form as $Lx = H(t)$. The virtue of this approach, besides an obvious economy in notation, is that it provides a conceptual setting for the study of systems of linear differential equations in which we can apply the known results for operator equations. However, as it stands (1.1) is too general to permit a systematic analysis leading to specific techniques of solution. Hence, in the sections which follow we should devote our efforts to the study of more specialized systems for which detailed information can be obtained.

Results from the general theory of first-order systems

We know that the first-order system is given as

$$\begin{matrix} x_1 = & a_{11}(t)x_1+\dots+ & a_{1n}(t)x_n+ & b_1(t) \\ \vdots & \vdots & \vdots & \vdots \\ x_n = & a_{n1}(t)x_1+\dots+ & a_{nn}(t)x_n+ & b_n(t) \end{matrix} \tag{1.2.1}$$

In which the a_{ij} (t) and $b_i\,(t)$ are continuous on an interval. In the matrix version (2.1)

$$X'=A(t)X+B(t) \tag{1.2.2}$$

$$A(t)=\begin{bmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \dots & a_{nn}\ (t) \end{bmatrix}$$

And

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, X^{`} = \begin{bmatrix} x_1 \\ \vdots \\ x'_n \end{bmatrix}, B(t) = \begin{bmatrix} b_1(t) \\ \vdots \\ b_n\ (t) \end{bmatrix}$$

As usual, initial-value problem for such a system requires that we find a solution X=X(t) of the system which satisfies an initial condition $X\,(t_0) = X_0$ where t_0 is a point in I and

$$X_0 = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is a point in R^n.

Systems of this type are especially important in the theory of linear differential equations because, among others, every normal nth-order linear differential equations can be transformedinto normal first-order system. We now prove this assertion as a theorem.

Theorem (1.2.1)[10]

Every normal nth-order linear differential equation is equivalent to an $n \times n$ system of normal first-order linear differential equations.

Proof.

Starting with

$$x^n + a_{n-1}\,(t)\,x^{(n-1)} + \dots + a_0\,(t)x = h\,(t) \tag{1.2.3}$$

Let $x_1 \dots x_n$ be new variables defined by

$$\begin{aligned} x_1(t) &= x(t) \\ 2(t) &= x^{`}(t) \\ &| \\ x_n(t) &= x^{(n-1)}(t) \end{aligned}$$

Then, (2.3) can be rewritten as

$$\begin{aligned} x_1 &= x_2 \\ x_2 &= x_3 \\ &| \\ x_{n-1} &= x_n \end{aligned} \tag{1.2.4}$$

$$\mathrm{x}_n = -a_0\,(t)\,x_1 - a_1\,(t)\,x_2 - \dots - a_{n\text{-}1}\,(t)\,x_n + h\,(t)$$

which is a system of the required kind. This theorem tells us that theory of the first-order linear systems includes the theory of nth-order linear equations as a special case. The converse, however, is not true because there exists first-order systems that cannot be converted into a single nth-order equation.

As in the case of a single nth-order equation, the theory of the first-order linear system is based on an existence and uniqueness theorem.

Theorem (1.2.2)[8]
Every initial-value problem

$X' = A(t)x + B(t), X(t_0) = X_0,$

Involving a normal $n \times n$ system of the first-order linear differential equations whose coefficients and right-hand sides are continuous on an interval I has a unique solution on I.
We now turn our attention to the homogeneous system

$$X' = A(t)X, \tag{1.2.5}$$

whose solution set is a subspace W of the vector space r_n defined in the preceding section. When combined with our earlier results on the dimension of the solution space of a normal homogeneous nth-order linear differential equation, theorem 1.2.1 suggests that W too is n-dimensional. Indeed, it is the proof follows easily from the lemma.

Lemma 1.2.3 (Hurvitz [1998])
Let $X_1, \ldots X <$ be solutions of $X = A(t) X I$, and t_0 be any point in I. Then, $X_1, \ldots, X_k$ are linearly dependent r_n if and only if the vectors $X_1(t_0)$ are linearly dependent in R^n.

$$\sum_{j=1}^{k} c_j X_j(t) = 0$$

For all t in I, then for $t = t_0$ we have,

$$\sum_{j=1}^{k} c_j X_j(t_0) = 0 \ ,$$

WHICH means that the vectors $X_1(t_0), \ldots X_k(t_0)$ are linearly dependent in R^n conversely suppose that

$$\sum_{j=1}^{k} c_j X_j(t_0) = 0$$

Where again, the c_j is constant and not all zero. Then, since the $X_j(t)$ is a solution of
$X' = A(t)X,$ the vector

$$X(t) = \sum_{j=1}^{k} c_j X_j(t)$$

Is a solution of the initial-value problem

$X' = A(t)X, X(t_0) = 0.$

However, by theorem (1.2.2), the only solution of this problem is the trivial solution X $(t) = 0$ for all t in I. Hence,

$$\sum_{j=1}^{k} c_j X_j(t_0) = 0$$

And $X_1, \ldots X_k$ are linearly dependent in r_w
To continue, we now prove

Theorem (1.2.4)[4]
If W is the solution space of the $n \times n$ system

$X' = A(t)X,$

Then, dim $W = n$

Proof.
In the first place, the dimension of W cannot exceed n for if it did we could find n+1 linearly independent vectors $X_1(t_0), X_{n+1}$ in W, then the vectors $X_1(t_0), X_{n+1}(t_0)$ are linearly independent in R^n which is impossible. To complete the proof, let E_i denotes the standard basis vectors in R^n, and let X_i be the solution of the initial-value problem.
Then, since $E_1,\ldots E_n$ are linearly independent in R^n Lemma (1.2.3) implies that $X_1,\ldots X_n$ are linearly independent in W. Thus, the dimension of W must be at least N and we are done.
From here, theory of the first-order system of n equations inunknowns develops in much the same way as the theory of a single nth-order equation. In particular, given n solutions

$$X_i(t) = \begin{bmatrix} x_{1i}(t) \\ \vdots \\ x_{ni}(t) \end{bmatrix}, 1 \le i \le n,$$

Of $X' = A(t)X$ on an interval I the vectors $X_1,\ldots X_n$ are a basis for the solution space of the equation if and only if their Wronskian.

$$W[X_1,\ldots X_n] = \begin{bmatrix} x_{11}(t) & \ldots & x_{1n}(t) \\ \vdots & & \vdots \\ x_{n1}(t) & \ldots & x_{nn}(t) \end{bmatrix} \tag{1.2.6}$$

Never vanishes on I The proof of this assertion depends on the fact that $W[X_1, \ldots X_n]$ is identically zero on I if and only if it vanishes at a single point of I. The details are not included in this work,
When $X' = A(t)X$ is the first-order system derived from the normal nth-order equation.

$$x^{(n)} + a_{n-1}(t)\,x^{(n-1)} + \ldots + a_0(t)\,x = 0 \tag{1.2.7}$$

$W[X_1, \ldots X_n]$ is no one other than the Wronskian of the n solution of (1.2.7), hence, the preceding result generalize known facts.
We conclude this section with a few remarks about solving non-homogenous first-order systems, which take the form

$$X' = A(t)X + B(t) \tag{1.2.8}$$

When expressed in matrix as we know, every solution of such a system can be written in the form $X_p + X_h$ where X_p is a particular solution of (1.2.8)
And X_k is a solution of the associated homogeneous system $X' = A(t)X$, thus if $X_1, \ldots X_n$ is a basis for the solution space of

$X' = A(t)X$

The general solution of (1.2.8) is

$X(t) = X_p(t) + c_1 X_1(t) + \ldots + c_n X_n(t).$

The c_1 being arbitrary constants. Moreover, just as in case of a single nth – order equation a particular solution X_p of (1.2.8) can be obtained from a basis $X_1,\ldots X_n$ of the associated homogeneous system by the method of variation of parameters. The procedure goes like this. First, form the vector

$$X(t)=\sum_{i=1}^{n} c_1(t)X_1(t) \tag{1.2.9}$$

and determine the c_1 (t) so that $X(t)$ is a solution of (1.2.8). Then, substitute (1.2.9) in (1.2.8) to obtain

$$\sum_{i=1}^{n} c_1(t)X'_i(t)+\sum_{i=1}^{n} c_1(t)X_1(t)=\sum_{i=1}^{n} c_1(t)A(t)X_i(t)+B(t)$$

Since X_i (t)= A (t) X_i (t) for i=1,…n the three equations above reduces to

$$\sum_{i=1}^{n} c_1(t)X_1(t)+=B(t),$$

Which in expanded form, reads

$$\begin{matrix} c'_1(t)x_{11}(t)+\ldots+ & c_n(t)x_{1n}(t)= & b_1(t) \\ \vdots & \vdots & \vdots \\ c_1(t)x_{n1}(t)+\ldots+ & c_n(t)x_{nn}(t)= & b_n(t) \end{matrix}$$

This system can be solved for a'_1 (t)…(n' (t)) since the determinant

$$W[X_1,\ldots X_n]=\begin{bmatrix} x_{11}(t) & \ldots & x_{1n}(t) \\ \vdots & & \vdots \\ x_{n1}(t) & \ldots & x_{nn}(t) \end{bmatrix}$$

First-order system with constant coefficients

The eigenvalue method is well suited to solving $n \times n$ systems of first-order linear differential equations with constant coefficients. Our discussion of these systems will divide into case that depends on the nature of the eigenvalues of the coefficient matrix (by which we mean the eigenvalues of the linear transformation defined by the matrix), just as the depend on the nature of the roots of their associated characteristic equation. In fact, as theorem (1.2.1) implies, the results we are about derive include our earlier work as a special case. We begin by considering the real eigenvalue of the coefficient matrix and, as usual, giving most of our attention to the homogeneous case. Thus, let

$$X' = A \tag{1.3.1}$$

be an $n \times n$ first-order linear system with coefficient matrix

$$A=\begin{bmatrix} \propto_{11} & \ldots & \propto_{1n} \\ \vdots & & \vdots \\ \propto_{n1} & \ldots & \propto_{nn} \end{bmatrix}$$

Lemma 1.3.1

For each real eigenvalue λ of A and each eigenvector

$$E_\lambda=\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$$

belonging to λ the function $X_\lambda = E_\lambda e^{\lambda t}$ is a solution of (1.3.2). Moreover, solutions formed in this way from distinct eigenvalues are linearly independent in $C(-\infty\infty)$.

Proof:
First of all $A E_\lambda = \lambda E_\lambda$ because E_λ is an eigenvector belonging to λ,
Thus,

$$A X_\lambda = \lambda E_\lambda e^{\lambda t} = \lambda X_\lambda$$

On the other hand, $X'_\lambda = \lambda E_\lambda e^{\lambda t} = \lambda X_\lambda$ and, $X'_\lambda = AX_\lambda$ as required.
Now let $\lambda_1, \ldots \lambda_k$ be distinct (real) eigenvalues of A with associated eigenvectors, $E'_{\lambda 1} = E_{\lambda k}$ and suppose that

$$c_1 E_{\lambda 1} e^{\lambda 1 t} + \ldots + \mathrm{c}_\mathrm{k} \lambda E_{\lambda k} e^{\lambda k t} = 0$$

Then, by setting $t = 0$ we have

$$c_1 E_{\lambda 1} e^{\lambda 1 t} + \ldots + c_k \lambda E_{\lambda k} e^{\lambda k t} = 0$$

and the linear independence of the $E_{\lambda i}$ in R^n implies $c_1 = \ldots = c_\mathrm{k} = 0$, hence, the $E_{\lambda k} e^{\lambda 1 t}$ are linearly independent in $C(-\infty\infty)$.
This result, together with theorem (2.4) immediately yields.

Theorem (1.3.2)
If A has n distinct (real) eigenvalues, $\lambda_1, \ldots \lambda_n$ and $E_{\lambda 1} \ldots E_{\lambda n}$ are eigenvectors belonging to these eigenvalues, then the general solution of the normal first-order system $X'\ AX$ is

$$X(t) = c_1 E\lambda_1 e^{\lambda 1 t} + \ldots + c_n \lambda E_{\lambda n} e^{\lambda n t}$$

Where, $c_1 \ldots c_n$ are arbitrary constants.

Example 1
Solve the system

$$x'_1 = x_1 + 3x_2 + \sin t$$

$$x'_2 = x_1 - x_2 - \cos t \tag{1.3.2}$$

Solution
A particular solution of this system can be obtained from the general solution of (1.2.1) by the method of variation of parameters described at the end of section (1.2.2). In this case, however, it is easier to use undetermined coefficients.
We seek a solution of the form below to (1.3.1)

$$x'_1 = A \sin t + A \cos t$$

$$x'_2 = C \sin t + D \cos t.$$

When we substitute these expressions in the given and collect like terms, we find that

$$(A + B + 3C + 1) \sin t + (-A + B + 3D) \cos t = 0$$

$$(A - C + D) \sin t + (B - C - D_{-1}) \cos t = 0$$

Since these equations must hold for all values of the coefficients *A*, *B*, *C*, and *D* must be chosen to make

$$\left.\begin{aligned} A+B+3C=-1 \\ -A+B+3D=0 \\ A-C+D=0. \\ B-C-D=1 \end{aligned}\right\}$$

The augmented matrix of this system of linear equation is

$$\begin{bmatrix} 1 & 1 & 3 & 0 & -1 \\ -1 & 1 & 0 & 3 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

And the row reduced echelon form of the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{1}{5} \\ 0 & 1 & 0 & 1 & \frac{2}{5} \\ 0 & 0 & 1 & 0 & -\frac{2}{5} \\ 0 & 0 & 0 & 1 & -\frac{1}{5} \end{bmatrix}$$

From this, we read the values of *A*, *B*, *C*, and *D* and conclude that

$$X_p(t)=\begin{bmatrix} -\frac{1}{5}\sin t+\frac{2}{5}\cos t \\ -\frac{2}{5}\sin t-\frac{1}{5}\cos t \end{bmatrix}$$

Is a particular solution of (1.3.2). The general solution of (1.3.1) therefore becomes

$$X(t)=-\begin{bmatrix} 3c_1e^{2t}-c_2e^{-2t}-\frac{1}{5}\sin t+\frac{2}{5}\cos t \\ c_1e^{2t}+c_2e^{-2t}\frac{2}{5}\sin t-\frac{1}{5}\cos t \end{bmatrix}$$

COMPLEX AND REPEATED EIGENVALUES

It remains for us to consider those cases in which the characteristic equation of the coefficient matrix for.

$X' = A\,X$

Has complex or repeated roots (or both). We begin with the complex case.

Complex eigenvalues

Since the entries of A are real by assumption, the characteristic equation of A is a polynomial equation with real coefficients, hence, if $\lambda = \propto + \beta_{\mathrm{i}} \beta \neq 0$, is an eigenvalue for A. The complex conjugate $\lambda' = a - \beta_i$ is also an eigenvalue of A. To find eigenvectors for these Eigenvalues, we extend the domain of the linear transformation A associated with A to include n–tipple of complex number as follows. Let C^n denotes the set of all column vectors.

$$Z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$

Where, the z_i are complex numbers and let addition and scalar multiplication in C^n be defined component wise;

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} y_1 & + & z_1 \\ \vdots & & \vdots \\ y_n & + & Z_n \end{bmatrix}, \quad z \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} zz_1 \\ \vdots \\ zz_n \end{bmatrix}$$

Where, in the second equation, z is an arbitrary complex z as arbitrary complex number.*
It is easy to verify that these definition convert C^n into a vector space provided complex number are used in place of real numbers as the scalars of the real of the original definition. In fact, once this change has been made, all the results previously established for R^n also hold for C^n. In particular, if

$$A = \begin{bmatrix} \propto_{11} \cdots & \propto_{11} \\ \vdots & \vdots \\ \propto_{n1} \propto_{nn} & \end{bmatrix}$$

is an $n \times n$ matrix with real or complex entries, then the mapping

$$\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \rightarrow \begin{bmatrix} \propto_{11} z_1 & + \ldots + & \propto_{1n} z_n \\ \vdots & & \vdots \\ \propto_{n1} z_1 & + \ldots + & \propto_{nn} z_n \end{bmatrix}$$

is a linear transformation that maps C^n to itself. Again a number λ (this time complex) is called an Eigenvalue for A and $Z \neq 0$ is called an eigenvector belonging to λ, if $AZ = \lambda Z$.

Lemma 2.1[3]
Let A be an $n \times n$ matrix with real entries and suppose that $\lambda = a + \beta_{\mathrm{i}}$, $\beta \neq 0$ is an eigenvalue for A then if Z is an eigenvector belonging to λ its complex conjugate Z is an eignvector belonging to the eigenvalue $\lambda = a + \beta_i$,

Proof:
Since $AZ = \lambda Z$
We must have

$$\overline{Az} = \overline{\lambda Z}$$

However, the entire of A are read so that $\bar{A} = A$ and the preceding equation reads

$$A\bar{z} = \overline{\lambda Z}$$

Recall that complex numbers are multiplied by the rule
$(a + bi)(c + di) = (ac - bd) + i(ad+bc)$
In short multiply as usual and the fact that $i^2 = -1$.

Lemma:2.2[2]

Let A be a real $n \times n$ matrix, and suppose that E_λ is an eigenvector in C^n of the complex eigenvalue $\lambda = \propto + \beta i$ *pf A* then. $E_\lambda e^{\lambda t}$ and $\bar{E}_\lambda e^{\bar{\lambda} t}$ are solution of equation $X' = A X$,

Proof:

The proof is the one we use before;

$$\frac{d}{dt} E_\lambda e^{\lambda t} = E_\lambda \frac{d}{dt} e^{(a+\beta i)} = E_\lambda \left(a + \beta i\right) e^{(a+\beta i)t} = E_\lambda e^{\lambda}$$

While $AE = A^{e\lambda t}$ thus

$$\frac{d}{dt} E_\lambda e^{\lambda t} = \bar{E}_\lambda \bar{\lambda}^t.$$

A similar argument can be given for the vector $E_\lambda e^{\bar{\lambda} t}$ *

Having come this far, we are now in a position to remove all references to complex numbers and complex-valued functions by invoking Euler's formula.[6]

$e^{(a+\beta i)} = e^a (\cos \beta + i \sin \beta)$,

For then

$$E_\lambda e^{\lambda t} = E_\lambda e^{at} \left(\cos \beta + i \sin \beta t\right)$$
$$\bar{E}_\lambda e^{\bar{\lambda} t} = \bar{E}_\lambda e^{at} \ (\cos \beta t - i \sin \beta t)$$

Moreover, since both of these functions are solutions of $X' = A X$, so are their linear combinations.

$$\frac{1}{2}\left(E_\lambda e^{\lambda t} + \bar{E}_\lambda e^{\bar{\lambda} t}\right) = e^{at} \left[\frac{E_\lambda + \bar{E}_\lambda}{2} \cos \beta t + \frac{i\left(E_\lambda + \bar{E}_\lambda\right)}{2} \sin \beta t\right]$$

and

$$\frac{i}{2}\left(E_\lambda e^{\lambda t} - \bar{E}_{\bar{\lambda} t}\right) = e^{a} \left[\frac{i(E_\lambda - \bar{E}_\lambda}{2} \cos \beta t - \frac{E_\lambda + \bar{E}_\lambda}{2} \sin \beta t\right].$$

Moreover, the coefficients that appear in this solution are real since

$$\frac{z + \bar{z}}{2} = a \, and \, \frac{z - \bar{z}}{2} = -b$$

Wherever $z = a + bi$ and $\bar{z} = a - bi$ are complex conjugates, thus we have proved.

Theorem 2.3:

Let $\lambda = a + \beta i$ be a complex eigenvalue for the $n \times n$ real matrix A and let E_λ be an eigenvector in C^n belonging to λ. Then, the functions

$X_1(t) = e^{at} (G_\lambda \cos \beta t + H_\lambda \sin \beta t)$

$X_2(t) = e^{at} (H_\lambda \cos \beta t - G_\lambda \sin \beta t)$, Where,

$$\frac{E_\lambda + \bar{E}_\lambda}{2} \text{ and } H_\lambda = \frac{i\,(E_\lambda - \bar{E}_\lambda)}{2}$$

are linearly independent solutions of $X' = A\,X$.
Strictly speaking we are allowing $X^{1} = AX$ having solution which is functions of a complex variable. As we remarked earlier, we can legitimately do so as soon as we have defined the notion of differentiability for such functions.

Example 2.1
Solve the first – order system

$$x'_1 = -x_2$$

$$x'_2 = x_1 \qquad (2.1)$$

Solution
In matrix form this system is

$$\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

And as in example (2.1). It has $\lambda = i$ as an eigenvalue.

$$E_i = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Is an eigenvector belonging to this system is

$$\frac{E_i + E_i}{2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \frac{i\,(E_i - E)}{2} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Thus, by theorem 2.4

$$X_1(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin t = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$

And

$$X_2(t) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$

Are linearly independent solutions of (2.2). The general solution of the system therefore is

$$X(t) = C_1 X_1(t) + C_2 X_2(t) = \begin{bmatrix} c_1 \sin t - c_2 \cos t \\ c_1 \cos t - c_2 \sin t \end{bmatrix}$$

Repeated eigenvalues. Except for language and notation, the result we have obtained so far for the first-order system with constant coefficients are identical with those we obtained earlier for the corresponding case involving constant coefficient nth-order equations. However, if the characteristic equation for A has a repeated root, something new happens.

Suppose for instance that λ_0 is a real eigenvalue of multiplicity two for A, by which we mean the $(\lambda-\lambda_0)^2$ is a factor of the characteristic equation for A. Then as we saw earlier, the system $X' = A X$ has a non-trivial solution of the form

$$X_1(t) = E\lambda_0 e^{\lambda 0t}$$

Where $E_{\lambda 0}$ is an eigenvector belonging to λ_0 our experience with n^{th}-order equation would suggest that the system also has a nontrivial solution of the form.

$$X_2(t) = B_{te} e^{\lambda ot} + C e^{\lambda ot} \tag{2.3}$$

Where, B and C are vectors in R^n but C is not necessarily a scalar multiple of when this happens the term $Ce^{\lambda ot}$, in (2.3) cannot be absorbed as part of $X_1(t)$ more generally, we have the following result.

Theorem 2.4[5]

Let λ_0 be a real; root of multiplicity m characteristic equation for the $n \times n$ matrix A. Then, the first-order system $X' = A X$ has m linearly independent solutions of the form.

$$X_1(t) = B_{11} e^{\lambda 0t}$$

$$X_2(t) = B_{21} te^{\lambda 0t} + B_{22} e^{\lambda 0t}$$

$$X_m(t) = B_{m1} e^{m-1} e^{\lambda ot} + B_{m2} t^{m-2} e^{\lambda 0t} + \ldots + B_{mm} e^{\lambda 0t}$$

Where, the B_{ij}^{ts} are vectors in R^n

An analogous result holds for repeated complex eigenvalues

Example 2.2

Find the general solution of

$$x'_1 = -2x_1 - 3x_2$$

$$x'_2 = 3x_1 + 4x_2 \tag{2.2}$$

Solution:

The characteristic equation for the coefficient matrix of this system is

$$\begin{vmatrix} -2-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = (\lambda-1)^2 = 0$$

Thus, $\lambda = 1$ is an eigenvalue of multiplicity two

To find an eigenvector belonging to λ, we seek a non-trivial solution of

$$\begin{bmatrix} -2 & -3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

Since this equation is satisfied if and only if $e_2 = -e_1$,

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Is an eigenvector for the system and

$$X_1(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$$

Is a solution. To continue, we seek a second solution of the form

$$X_2(t) = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} te^t + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} e^t$$

When this expression is substituted into (2.4) and like term is collected, we find that.

$(3B_1 + 3B_2)\, te^t + (B_1 + 3C_1 + 3C_2)\, e^t = 0$

$(3B_1 + 3B_2)\, te^1 + (-B_2 + 3C_1 + 3C_2)\, e^t = 0.$

These equations imply that

$$B_2(t) = -B_1 \; and \; C_2 = -\frac{1}{3} B_1 - C_1$$

Where B_1 and $<C_1$ are arbitrary. For instance, if we set $B_1 = 3$ and $C_1 = 0$, then

$$X_2(t) = \begin{bmatrix} 3 \\ -3 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^t$$

Thus, the general solution of (2.4) can be written as

$$X(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 3 \\ -3 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^t \right)$$

APPLICATIONS

Here, we present two applications of linear differential equations to biological and physical problems as seen below.

An application to biology

In biology and medicine, one is often interested in describing how a chemical compound such as drug accumulates in cells as the compound diffuses across cell walls. One model that can be used to approximate this process consists of a sequence of *n* compartments or boxes which corresponds to the cells in which the compound is accumulating and which are arranged linearly as shown in Figure 5, we propose to derive equations that describe the diffusion process under the following assumptions.

1. The distribution of the compound within each cell is uniform at all times.
2. The rate of diffusion of the compound from *cell* $i-1$ to *cell* (amount per unit time per unit of cell wall area shared) is a constant, $\propto_i -1i$ times the concentration of the compound (amount per unit volume) in *cell* $i-1$ similarly, the rate of diffusion from cell back 1 to *cell* $i-1$ is a constant, $\propto_i -1i$ times the concentration in *cell i*.
3. The volume of each cell is equal to the area of each wall of the cell across which the compound is being diffused. (this would be the ease, i.e., if the cell were assumed to be unit cubes with shared faces)

With these assumptions in force, we let $C_i = C_i(t)$ denote the concentration of the compound in that time *t* then as shown in Figure 1 diffusion across the walls between *cell* $i-1$ and $i+1$ *causes* C_1 to increase at the rate.

$$\propto_{i-1-i} + \propto_{i11'i} C_{i\,1\,i'} \tag{3.1}$$

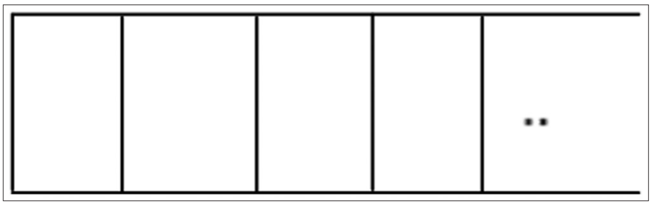


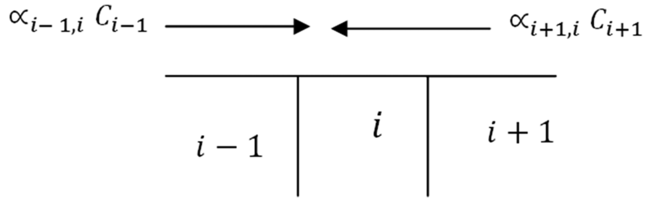
Figure 1: Diffusion across the walls

And decrease at the rate

$$(\propto_{ii\text{-}1} + \propto_{ii+,1})\ C_i \tag{3.2}$$

This expression also applies to the cell at the end of the sequence if we set

$\propto_{0,1} = \propto_{1}, 0 \propto_{nn+1} \propto_{n+1,n} = 0$



If follows from (5.1) and (5.2) that the rate of change of the concentration in the i^{th} cell is

$$\frac{dC_i}{d <} = \propto_{i-1,i} C_{i-1} - \left(\propto_{i,i-1} + \propto_{i,i+1}\right) C_i + \propto_{i+1,i} C_{i+1} \tag{3.3}$$

Thus, the diffusion process is described mathematically by a system of nth first-order constant coefficient linear differential equations in n unknowns.
In the interest of further simplicity w now assume that we are confronted with only two types of cell, which alternate in sequence and which have respective diffusion rates ∝ and β. Then, if the sequence starts with a cell whose diffusion rate is ∝ (3.2) becomes

$$\frac{dC_1}{dt} = -\propto C_1 + \beta C_2$$

$$\frac{dC_2}{dt} = \propto C_1 - 2\beta C_2 + \beta C_3$$

$$\frac{dC_3}{dt} = \beta C_2 - 2\propto C_3 + \beta C_4$$

$\vdots$

In actual practice, the number of cells in the sequence is usually large, and (3.3) must be solved on a computer. Nevertheless, it is instructive to consider this system for small values of n, if only to determine whether the solutions conform to our expectations of what ought to happen.
The two-cell model. Suppose that the sequence consists of only two cells and a unit amount of the compound is injected into the first cell at time $t = 0$ the concentration in the second cell being zero. In this (3.3) reduces to the 2 × 2 system.

$$\begin{aligned} \frac{dC_1}{dt} &= -\propto C_1 + \beta C_2 \\ \frac{dC_2}{dt} &= \propto C_1 - \beta C_2 \end{aligned} \tag{3.4}$$

And is subject to the initial condition

$C_1(0) = 1, \quad C_2(0) = 0$

The characteristic equation of the coefficient matrix of (5.4) is

$\lambda^2 + (\propto + \beta)\,\lambda = 0$

And the eigenvalues for the system therefore are

$\lambda = 0$ and $\lambda = -(\propto + \beta)$

A routine calculation reveals that

$$E_0 + \begin{bmatrix} 1 \\ \frac{\propto}{\beta} \end{bmatrix} and\ E_{-(\propto + \beta)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

are eigenvectors belonging to these eigenvalues. It follows that the general solution of (5.4) is

$$\left.\begin{aligned} C_1(t) &= A + Be^{-(\propto + + \beta)t} \\ C_2(t) &= \frac{\propto}{\beta} A - Be^{-(\propto + \beta)t} \end{aligned}\right\}$$

Finally, the given initial conditions imply that

$$A = \frac{\beta}{\propto + \beta}\ and\ B = \frac{\propto}{\propto + \beta}$$

and hence that

$$C_1(t) = \frac{\beta}{\propto + \beta} + \frac{\propto}{\propto + \beta} e^{-(\propto + \beta)t}$$

$$C_2(t) = \frac{\beta}{\propto + \beta}\left(1 - e^{-(\propto + \beta)t}\right)$$

Note that as $t \to \infty$, $C_1(t)$ and $C_2(t)$ approach the steady-state values,

$$C_1 = \frac{\beta}{\propto + \beta} C_2 = \frac{\propto}{\propto + \beta}$$

Moreover, in the steady state, the concentrations in cell 1 and 2 are in the ratio $\frac{\beta}{\propto}$ both these results are just what one would expect.

An Application to Physics

Imagine two masses m_1 and m_2 coupled as shown in Figure 2 and constrained to vibrate horizontally. We seek equations of motion for this system under the assumptions that it is immersed in a viscous medium which gives rise to retarding forces directly proportional to the velocities of the masses and that each mass is subject to an external force that varies with time.

We begin our analysis be letting l_1, l_2 and i_3 denote the natural lengths of the spring s as they appear from left to right in the figure, and k_1, k_2 and k_3 their respective spring constants. There are three kinds of forces acting on the first mass:

1. Restoring forces $k_1\,(x_1 - x_0 - l_1)$ and $(x_2 - x_1 - l_2)$ [Figure 3]
2. A retarding force $a_1, \dfrac{dx_1}{dt}, a_1$ a constan
3. As external force $F_1 = F_1\,(t)$.

Similar forces act on the second mass, and therefore, Newton's second law implies that the motion of the system is governed by the pair of second-order linear differential equations.

$$m_1 \frac{d^2x_2}{dt^2} = -k_1\left(x_1 - x_0 - l_2\right) + k_2\left(x_2 - nx_2 - l_1\right) - a_1\frac{dx_1}{dt} + F_1\ (t)$$
$$m_2 \frac{d^2x_2}{dt^2} = -k_1\left(x_2 - x_1 - l_2\right) + k_2\left(x_3 - x_2 - l_3\right) - a_2\frac{dx_2}{dt} + F_2\ (t) \tag{3.2.1}$$

In the interest of simplicity, we now assume that,

$l_1 = l_2 = l_3 = 1$ and $k_1 = k_2 = k_3 = k$

Then, (3.2.1) can be rewritten as

$$m_1 \frac{d^2x_1}{dt^2} + a_1\frac{dx_1}{dt} - k\left(x_0 - 2x_1 + x_2\right) = F_1\ (t)$$
$$S_2 \frac{d^2x_2}{dt^2} + a_2\frac{dx_2}{dt} - k\left(x_1 - 2x_2 + x_3\right) = F_2\ (t) \tag{3.2.2}$$

Up to this point, we have allowed x_0 and x_3 to vary with time. We now assume, however, that they are fixed with $x_0 = A_0$ and $x_2 = A_2$. Then, when F_1 and F_2 are identically zero, (3.2.2) has the unique time-independent solution.

$$x_1 = \frac{2}{3}A_0 + \frac{1}{3}A_3$$
$$x_2 = \frac{1}{3}A_0 + \frac{2}{3}A_3 \tag{3.2.3}$$

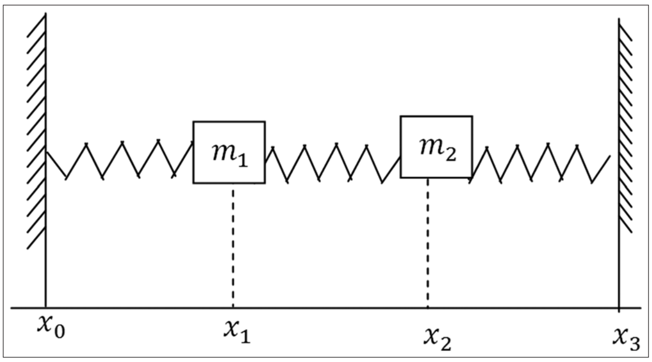


Figure 2: The unique time-independent solution $[X_0 - X_3]$

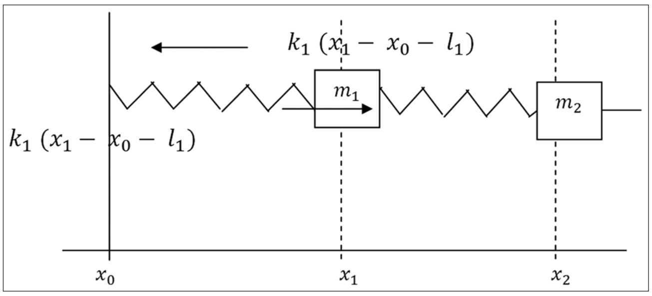


Figure 3: The unique time-independent solution $[X_0 - X_2]$

Found by setting the derivative in each equation equal to zero and solving for x_1 and x_2. Thus, in the absence of external forces and moving boundaries the spring–mass system is in equilibrium with the masses equally spaced between the walls that make a change of variables and measure displacements form these equilibrium positions which we now denote by A_1 and A_2 respectively.[9] Hence, we set

$$y_1 = x_1 - A_1 \; and \; y_2 = x_2 - A_2$$

In addition, we introduce the momentum variables

$$y_3 = m_1 \frac{dx_1}{dt} \; and \; y_4 = m_2 \frac{dx_2}{dt}$$

Thereby, converting (3.2.2) into the first-order system

$$\begin{aligned}
\frac{dy_1}{dt} &= \frac{1}{m_1} y_3 \\
\frac{dy_2}{dt} &= \frac{1}{m_1} y_4 \\
\frac{dy_3}{dt} &= -2k_{y1} + ky_2 - \frac{a_1}{m_1} y_3 + F_1(t) \\
\frac{dy_4}{dt} &= -ky_1 - 2k_{y2} - \frac{a_2}{m_2} y_4 + F_2(t)
\end{aligned}$$

This system can be written in matrix form as

$$Y^{1} = Ay + B \tag{3.2.4}$$

Where

$$A = \begin{bmatrix} 0 & 0 & \frac{1}{m_1} & 0 \\ 0 & 0 & 0 & \frac{1}{m_2} \\ -2k & k & \frac{-a_2}{m_2} & 0 \\ k & -2k & 0 & \frac{-a_2}{m_2} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ F_1(t) \\ F_2\ (t) \end{bmatrix}$$

The characteristic polynomial of A is

$$\begin{aligned}
&\lambda^4 + \left(\frac{a_1}{m_1} + \frac{a_2}{m_2}\right)\lambda^3 + \left[\frac{a_1 a_2}{m_1 m_2} + 2k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\right] \\
&\quad + \frac{2k}{m_1 m_2}(a_1 + a_2)\lambda + \frac{3k^2}{m_1 m_2}.t^2
\end{aligned} \tag{3.2.5}$$

We now simplify the problem further by assuming that the viscous forces acting on the masses can be ignored. Then, $a_1 = a_2 = 0$, and the characteristic equations of A become.

$$\lambda^4 + 2k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\lambda^2 + \frac{3k^2}{m_1 m_2} = 0$$

We find that

$$\lambda^2 = k\ \ [-(\mu_1 + \mu_2) \pm \sqrt{(\mu_1 + \mu_2)^2 - 3\mu_1\mu_2}]$$

Where,

$$\mu_1 = \frac{1}{m_1}\mu_2 = \frac{1}{m_2}$$

And,

$(\mu_1+\mu_2)^2-3\mu_1\ \mu_2>0$

Hence, the eigenvalues for this problem have the form

$\lambda_1 = (t)i,\ \lambda_2 = -(t)\ i$

$\lambda_3 = vi,\ \lambda_4 = -vi,$ (3.2.6)

With $(t)>0$ and $v>0$ to find eigenvectors for these eigenvalues, we must find a non-trivial solution of

$$\begin{bmatrix} 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & \mu_2 \\ 2k & k & 0 & 0 \\ -k & -2k & 0 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \lambda_i \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

As λ_1 assumes each of the values λ_1, λ_2, λ_3, λ_4 we omit the computational details and simply assert that vectors

$$\begin{bmatrix} k \\ 2k + \frac{\lambda_i^2}{\mu_1} \\ \frac{k\lambda_i}{\mu_1} \\ \frac{\lambda_i}{\mu_2}\ (2k + \frac{\lambda_i^2}{\mu_1}) \end{bmatrix} \quad (3.2.7)$$

Satisfy the system. Thus, when there are no viscous forces, the homogeneous equation,

$Y' = AY,$

Has the four linearly independent solutions

$Y_i\ (t) = E_{\lambda i}\ e^{\lambda it}$

And,

$$y_1(t) = A_1 k \cos t(v)t + A_2 k \sin(i)t + A_3 k x = \cos tv\, vt + A_\$ \sin vt$$

$$y_2(t) = A_1\left(2k - \frac{(i)^2}{u_1}\right)\cos(i) + A_2\left(2k - \frac{(i)^2}{\propto_1}\right)\sin(i)t$$

$$A_3\left(2k - \frac{v^2}{u_1}\right)\cos vt + A_4\left(2k - \frac{v^2}{\propto_1}\right)\sin vt,$$

REFERENCES

1. Breuer F, Noel J. Ordinary Differential Equations; A First Course. New York: W.A Benjamin; 1961.
2. Codington E. Anintroduction to Ordinary Differential Equations. Englewood Cliffs, New Jersey: Prentice Hall; 1955.
3. Codington E, Levinson N. Theory of Ordinary Differential Equations. New York: McGraw-Hill; 1955.
4. Ford L. Differential Equations. 2nd ed. New York: McGraw-Hill; 1995.
5. Gelfand I. Lectures on Linear Algebra. New York: Inter-Science; 1961.
6. Golomb M, Shanks M. Elements of Ordinary Differential Equations. 2nd ed. New York: McGraw-Hill; 1965.
7. Hurwitz W. Lectures on Ordinary Differential Equations. New York, Cambridge, Massachusetts: M.I.T. Press, Wiley; 1958.
8. Inca E. Ordinary Differential Equations. New York: Dover; 1956.
9. Kaplan W. Ordinary Differential Equations. Reading, Massachusetts: Addison Wesley; 1958.
10. Leighton W. Ordinary Differential Equations. 3rd ed. Belmont, California: Wadsworth; 1970.
11. Lin CC, Segal L. Mathematics Applied to Deterministic Problems in the Natural Sciences. New York: Macmillan; 1974.