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# On Extendable Sets in the Reals (R) With Application to the Lyapunov Stability Comparison Principle of the Ordinary Differential Equations

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## ABSTRACT

This work produces the authors' own concept for the definition of extension on R alongside a basic result he tagged the basic extension fact for R. This was continued with the review of existing definitions and theorems on extension prominent among which are the Urysohn's lemma and the Tietze extension theorem which we exhaustively discussed, and in conclusion, this was applied extensively in resolving proofs of some important results bordering on the comparison principle of Lyapunov stability theory in ordinary differential equation. To start this work, an introduction to the concept of real numbers was reviewed as a definition on which this work was founded.

**Key words:** Compact set, comparison principle, normal space, regular space, the set-R, Tietze's extension theorem

## INTRODUCTION

It is a well-known fact that the set of real Numbers is closed with the basic operations of addition and multiplication in view of this, we develop the basic elementary topology of the set using the following fundamental definitions and theorems.

## THE REALS (THE SET, R)

The reals or the set R is a field which is a non-empty set that has the following properties.

A. To every pair $x, y$ of scalars, there exists a corresponding scalar $x+y$ called the sum of $x$ and $y$ such that
   i. Addition is commutative
   ii. Addition is associative
   iii. There exists a unique number or scalar 0 such that $x+0=x$
   iv. For every element in the field, there exists an element $(-x) \in F$ such that $x+(-x)=0$.

B. To every pair $x$ and $y$ of scalars, there corresponds a scalar $xy$ called the product of $x$ and $y$ such that
   i. Multiplication is commutative
   ii. Multiplication is associative
   iii. There exists a unique non-zero scalar 1 such that $x*1=x$
   iv. To every non-zero scalar $x$, there corresponds a unique scalar $\frac{1}{x}$ such that $x*\frac{1}{x}=1$.

C. Multiplication is distributive under addition, i.e., $x(y+z)=xy+xz$.

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### Definition 1.1[1]

Any subset of R which satisfies the conditions of an extension in R as given in the definition below is said to be an extendable subset of R,

### Definition 1.2[2]

Let $X=\mathbb{R}$ be a normal space (in topological sense) and let $A$ be any closed subspace of R. $A$ map $f: A \rightarrow [a,b], [a,b]=\mathbb{N}$ is called an extension of $A$ if

i. $f$ is continuous

ii. $f: A \rightarrow [a,b]=\mathbb{N}$ such that $f=\overline{f}$.

**Theorem 1.1 (Basic Extension Fact for R)[3]**

Every compact (closed and bounded) subset of R is extendable, but non-compact subsets are non-extendable (non-continuable)

**Proof:** Let $S$ be the compact subset in R. Then, for any subset, there exists condition such that every sequence in $S$ has a subsequence that converges to a point in $S$.

Let $S$ be compact and let $\{X_n\}$ be an arbitrary sequence in $S$ and since any sequence in $S$ has a convergent subsequence, $\{X_{n_k}\}$ has convergent subsequences. Let that subsequence. $\{X_{n_k}\} \to x^*$ and since any convergent sequence in $S$ has its limit in $S$, $x^* \in S$.

Now, we show that $S$ is extendable by definition (1.2). Let $\mathbb{R}$ be a normal space. Since $S$ is compact, $S$ is a closed subset of R.

Hence, we define the map $f : S \to [a,b] = \mathbb{N}$ and show that

(i) $f$ is continuous

(ii) $f: S \to [a,b] = \mathbb{N}$ is such that $f = \bar{f}$

(iii) Let $f : S \to \mathbb{R}$ be a function and we define the function $f(x_n)$ such that $f(x_n) \to f(x)$, for each $x_n \to x$. Then, $f$ is continuous and to show (ii) since $S$ is a compact subset of $\mathbb{R}$ which is also a normal space, then $S$ is a closed set where $f$ $(s_1)$ = $a$ and $f(s_n) = b$ natural numbers since the image of a closed set is closed, i.e., $f^{-1} \in [a,b]$ is closed in $S$ for closed set $[a,b]$ in $\mathbb{N}$ then $f = \bar{f}$. Hence, the compact subset of $\mathbb{R}$ is extendable because it satisfies the conditions of definition (1.2).

(iv) To prove that a non-compact subset is non-extendable, we pick $S = (a,b)$ or $[b,\infty]$ say, a non-compact subset and show $f$: $S \to [a,b]$ such that

i. $f$ is continuous and

ii. $f$: $S \to [a,b] = \mathbb{N}$ and $f = \bar{f}$

This is obvious because though any $f$ defined on these sets may be continuous in the set $S$, they will not satisfy condition (ii) since those sets are not compact (or closed) and as such no $f$ can operate in any non-compact set to secure extension.

## RESULTS ON EXTENSION $\mathbb{N}$ $\mathbb{R}$

**Definition 2.1[4]**

Given that, one point sets are closed in the set $X$, then $X$ is said to be regular if for each pair consisting of a point $x$ and a closed set $B$ disjoint from $x$ such that there exist disjoint open sets containing $x$ and $B$, respectively. On the other hand, the space $X$ is said to normal if for each pair $A, B$ of disjoint closed sets of $X$, there exist disjoint open sets containing $A$ and $B$, respectively. Hence, a regular space is Hausdorff and a normal space is regular $A$ space $X$ is said to be completely normal if every subspace of $X$ is normal.

**Lemma 2.1[5]**

Let $X$ be a topological space and let one point sets in $X$ be closed.

(a) $X$ is regular if and only if given a point $x \in X$ and a neighborhood $U$ of $x$ there is a neighborhood $V$ of $x$ such that $\bar{V} \subset U$.

(b) $X$ is normal if and only if given a closed set $A$, and an open set $U$ containing $A$, there is an open set $V$ containing $V$ such that $\bar{V} \subset U$.

**Proof:** (a) Suppose that $X$ is regular and suppose that the point $x$ and the neighborhood $U$ of $x$ are given. Let $B = X - U$; then, $B$ is a closed set. By hypothesis, there exist disjoint open sets $V$ and $W$ containing $x$ and $B$, respectively. The set $\bar{V}$ is disjoint from $B$ since if $y \in$ B, the set $W$ is a neighborhood of $y$ disjoint from $V$. Therefore, $\bar{V} \subset U$ as desired.

To prove the converse, suppose the point $x$ and the closed set $B$ not containing $x$ are given. Let $U = X - B$.

By hypothesis, there is a neighborhood $V$ of $x$ such that $\bar{V} \subset U$. The open sets $V$ and $X - V$ are disjoint open sets containing $x$ and $B$, respectively. Thus, $X$ is regular.

(b) This proof uses exactly the same argument; one just replaces the point $x$ by the set $A$ throughout.

**Theorem 2.2[6]**

a. Every regular space with a countable basis is normal
b. Every metrizable space is normal
c. Every compact Hausdorff space is normal
d. Every well-ordered set $X$ is normal in the order topology.

**Proof**: (d) Let $X$ be a well-ordered set. We assert that every interval of the form $(x, y)$ is open in $X$. If $X$ has a largest element and $y$ is the element, $(x, y)$ is just a basic element about y. If $y$ is not the largest element of $X$, then $(x, y)$ equals the open set $(x, y')$ where $y'$ is the immediate successor of $y$.

Now, let $A$ and $B$ be disjoint closed sets in $X$; assume for the moment that neither $A$ nor $B$ contains the smallest element $a_0$ of $X$.

For each $a \in A$, there exists a basis element about $A$ disjoint from $B$; it contains some interval of the form $(x, a)$ (Here is where we use the fact that is not the smallest element of $X$) choose, for each $a \in A$, such an interval $(x_a, a)$ disjoint from $B$. Similarly, for each $b \in B$, choose an interval $(y_0, b)$ disjoint from $A$.

The sets

$U = U_{a \in A}(x_a, a)$ and $V = V_{b \in B}(y_b, b)$

are open sets containing $A$ and $B$, respectively; we assort they are disjoint for supposing that $z \in U \cap V$.

Then, $z \in (x_a, a) \cap (y_b, B)$ for some $a \in A$ and some $b \in B$.

Assume that $a < b$. Then, if $a \le y_b$, the two intervals are disjoint, while if $a < y_b$, we have $a \in (y_b, b)$, contrary to the fact that $(y_b, b)$ is disjoint from $A$. Similarly, contradiction occurs if $b < a$.

Finally, assume that $A$ and $B$ are disjoint closed set in $X$, and $A$ contains the smallest element $a$ of $X$. The set $\{a_0\}$ is both open and closed in $X$. By the result of the preceding paragraph, there exist disjoint open sets $U$ and $V$ containing the closed sets $A-\{a_0\}$ and $B$, respectively. Then, $Uv\{a_0\}$ and $V$ are disjoint open sets containing $A$ and $B$, respectively.

**Theorem 2.2 (The Urysohn's Lemma)[7]**

Let $X$ be a normal space let $A$ and $B$ be disjoint closed subsets of $X$. Let $[a,b]$ be closed interval in the real line. Then, there exists a continuous map

$f : X \to [a,b]$

Such that $f(x) = a$ for every $x$ in $A$, and $f(x) = b$ for every $x$ in $B$

We now state and prove the very important consequence of the Urysohn's lemma and that is the:

**Theorem 2.3 (Tietze Extension Theorem)[8-10]**

Let $X$ be a normal space; $A$ be a closed subspace of $X$.

a. Any continuous map of $A$ into the closed interval $[a,b]$ of $\mathbb{R}$ may be extended to a continuous map of all of $X$ into $[a,b]$.
b. Any continuous map of $A$ into $\mathbb{R}$ may be extended to a continuous map of all of $X$ into $\mathbb{R}$.

**Proof:** The idea of this proof is to construct a sequence of continuous functions $S_n$ defined on the entire space $X$ such that the sequence $S_n$ converges uniformly and such that the restriction of $S_n$ to $A$ approximates *f* more closely as $n$ becomes large. Then, the limit function will be continuous and its restriction to $A$ will equal *f*.

**Step 1**: The first step is to construct a particular function *g* defined on all of $X$ such that $g$ is not too large and such that $g$ approximates *f* on the set $A$ to a fair degree of accuracy. To be more precise, let us take the case $f : A \to [-r, r]$. We assert that there exists a continuous function $g : X \to \mathbb{R}$ such that

$|g(x)| \le \frac{1}{3} r$ for all $x \in X$,

$|g(a) - f(a)| \le \frac{2}{3} r$ for all $a \in A$,

The function $g$ is constructed as follows:

Divide the interval $[-r, r]$ into three equal intervals of length $\frac{2}{3} r$:

$$I_1 = \left[-r, -\frac{1}{3} r\right],\ I_2 = \left[-\frac{1}{3} r, \frac{1}{3} r\right],\ I_3 = \left[\frac{1}{3} r, r\right]$$

Let $B$ and $C$ be the subsets

$B = f^{-1}(I_1)$ and $C = f^{1}(I_3)$

a. Because *f* is continuous, $B$ and $C$ are closed disjoint subsets of $A$. Therefore, they are closed in $X$. By the Urysohn's lemma function, there exists a continuous function

$$g : X \to \left[-\frac{1}{3} r, \frac{1}{3} r\right]$$

having the property that $g(x) = -\frac{1}{3} r$ for each $x$ in $B$ and $g(x) = \frac{1}{3} r$ for each $x$ in $C$.

Then, $|g(x)| \le \frac{1}{3} r$ for all $x$. We assert that for each $a$ in $A$,

$$|g(a) - f(a)| \le \frac{2}{3} r.$$

There are three cases. If $a \in B$, then both $f(a)$ and $g(a)$ belong to $I_1$. If $a \in C$, then $f(a)$ and $g(a)$ are in $I_3$. And if $a \notin B \cap C$, then $f(a)$ and $g(a)$ are in $I_2$. In each can $|g(a) - f(a)| < \frac{2}{3} r$ [Figure 1].

**Step 2:** We now prove part (a) of the Tietze theorem. Without loss of generality, we can replace the arbitrary closed interval $[a,b]$ of $\mathbb{R}$ by the interval $[-1,1]$.

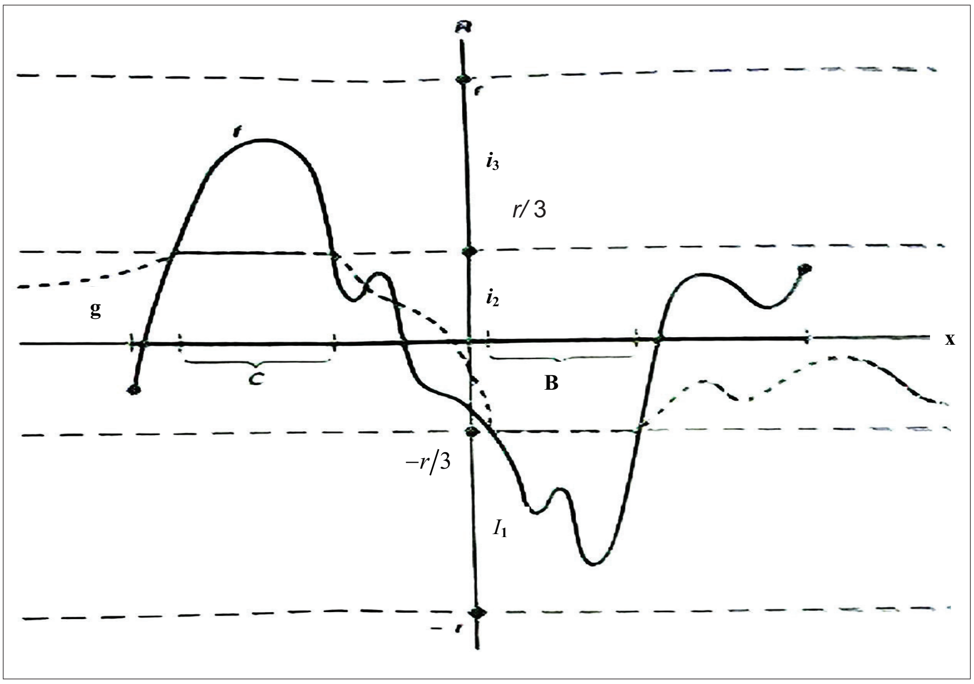


**Figure 1:** Countability and separation axioms

Let $f: X \to [-1,1]$ be a continuous map. Then, $f$ satisfies the hypothesis of step 1, with $r = 1$. Therefore, there exists a continuous real-valued function defined on all of $X$ such that

$$|g(x)| \le \frac{1}{3} \quad \text{for } x \in X,$$

$$|f(a) - g'(a)| \le \left(\frac{2}{3}\right) \quad \text{for } a \in A,$$

Now consider the function $f - g_1$. This function maps $A$ into the interval $\left[-\frac{2}{3}, 2\right]$ so we can apply step1 again, letting $r = \frac{2}{3}$. We obtain a real-valued function defined on all of $X$ such that

$$|g(x)| \le \frac{1}{3}\ (\frac{2}{3}) \quad \text{for } x \in X,$$

$$|f(a) - g_1(a)g_2| \le \left(\frac{2}{3}\right)^2 \quad \text{for a} \in A,$$

Then, we apply step 1 to the function $f$ - $g_1 - \cdots - g_n$ and so on.
At the general step, we have real-valued functions $f -\ g_1, \ldots, g_2$ defined on all of $X$ such that

$$|f(a) - g(a) - \cdots - g_n(a)| \le \left(\frac{2}{3}\right)^n \text{ n}$$

For $a \in A$. Applying step 1 to the function $-g_1 - \cdots -\ g_n$, with $r = \left(\frac{2}{3}\right)^n$, we obtain a real-valued function $g_{n+1}$ defined on all of $X$ that

$$|g_{n+1}(x)| \le \frac{1}{3}\left(\frac{2}{3}\right)^n \quad \text{for x} \in X,$$

$$|f(a) - g_1(a) - \cdots - g_{n+1}(a)| \le \left(\frac{2}{3}\right)^{n+1} \quad \text{for } a \in A,$$

By induction, the functions $g_n$ are defined for all $n$.
We now define

$$g(x) = \sum_{n=1}^{\infty} g_n(x)$$

For all $x$ in $X$. Of course, we have to know that this infinite series converges. But that follows from the comparison theorem of calculus; it converges by comparison with the geometric series.

$$\frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1}$$

To show that $g$ is continuous, we must show that the sequence $S_n$ converges to $g$ uniformly. This fact follows at once from the "Weierstrass *M*-test" of analysis. Without assuming this result, one can simply note that if $k > n$, then

$$|s_k(x) - s_n(x)| = \left|\sum_{i=n+1}^{k} g_i(x)\right| \le$$

$$\frac{1}{3}\sum_{i=n+1}^{k}\left(\frac{2}{3}\right)^{i-1} < \frac{1}{3}\sum_{i=n+1}^{\infty}\left(\frac{2}{3}\right)^{i-1} = \left(\frac{2}{3}\right)^n$$

Holding n fixed and letting $\to \infty$, we see that

$$|g(x) - s_n(x)| \le \left(\frac{2}{3}\right)^n$$

For all $x \in X$. Therefore, $S_n$ converges to $g$ uniformly. We show that $g(a) = f(a)$ for $a \in A$.

let $S_n(x)=\sum_{i=1}^{n} g_i(x)$, the nth partial sum of the series. Then, $g(x)$ is by definition the limit of the infinite sequence $S_n(x)$ partial sums. Since for all $a$ in $A$, it follows that $S_n(a)\to f(a)$ for all $a\in A$; therefore, we have $f(a)$ for $a\in A$.
Finally, we show that $g$ maps $X$ into the interval $[-1,1]$ this condition is, in fact, satisfied automatically since the series $\frac{1}{3}\sum\left(\frac{2}{3}\right)^n$ converges to 1. However, this is just a lucky accident rather than an essential part of the proof. If all we knew were that $g$ mapped $X$ into $\mathbb{R}$, then the map $r°g$, where $r:\mathbb{R}\to[-1,1]$ is the map

$r(y)=y$ if $|y|\le 1$,

$r(y)= y/|y|$ if $|y| \ge 1$,

Would be an extension of *f* mapping $x$ into $[-1,1]$.

**Step 3:** We now prove part (b) of the theorem, in which *f* maps $A$ into $\mathbb{R}$, we can replace $\mathbb{R}$ by the open interval $(-1,1)$ since this interval is homeomorphic to $\mathbb{R}$.
Hence, let *f* be a continuous map from $A$ into $(-1,1)$. The half of the Tietze theorem already proved shows that we can extend *f* to a continuous map $g:X\to[-1,1]$ mapping $X$ into the closed interval. How can we find a map *h* carrying $X$ into the open interval?
Given $g$, let us define a subset $D$ of $X$ by the equation
$D=g^{-1}(\{-1\})\cup g^{-1}(\{1\})$.
Since $g$ is continuous, $D$ is closed a subset of $X$. Because $g(a)=f(a)$ which is contained in $(-1,1)$ the set $A$ is disjoint from $D$. By the Urysohn's lemma, there is a continuous function $\varnothing: X\to[0,1]$ such that $\varnothing(D)$ = $\{0\}$ and $\varnothing(A)$ = $\{1\}$. Define

$h(a)=\varnothing(x)g(x)$

Then, *h* is continuous, being the product of two continuous functions. Furthermore, *h* is extension of $f$ since for $a$ in $A$,

$h(a)=\varnothing(a)g(a)=1g(a)=f(a)$

Finally, *h* maps all of $X$ into the open interval $(-1,1)$. For if $x\in D$, then $h(x)°g(x)=0$. Moreover, if $x\notin D$, then $|g(x)|<1$; it follows that $h(x)\le 1-\|g(x)\|$.

## APPLICATION OF EXTENDABLE SETS IN $\mathbb{R}$ TO THEORY OF THE COMPARISON PRINCIPLE IN LYAPUNOV STABILITY

Let $x(t), t\in(t_0,t_1), 0\le t_0 < t_1 < °$ be a solution of the given system of ordinary differential equations

$X' = F(t,x)$ (3.1)

Then, $x(t)$ is extendable to the point $t=t_1$ if and only if it is bounded on $(t_0,t_1)$. Again if $x(t), t\in(t_0,T_1)$, $0\le t_0 < T \le +\infty$ be a solution of the system (3.1). Then, $x(t)$ is said to be "non-continuable" or "non-extendable" to the right if *T* equals $+\infty$ or if $x(t)$ cannot be continued to the point T. Moreover, every extendable to the right solution of the system 3.1 is part of a non-continuable to the right solution of the same system.

### Theorem 3.1

Let $x(t), t\in(t_0,t_1)$ $t_1 \ge t_0$, an extendable to the right solution of the system (3.1), then there exists a non-continuable to the right solution (3.1) which extends $x(t)$ that is a solution $y(t),(t_0,t_2)$ such that $t_2 > t_1$, $y(t), t\in(t_0,t_1)$ and $y(t)$ is non-continuable to the right (here, $t_2$ may be equal to $+\infty$).

**Proof:** It suffices to assume that $t_0>0$. Let $Q=(0,\infty)\times\mathbb{R}^n$ and for $m=1,2,\ldots,$ let

$$Q_m=\left((t,u)\in Q; t^2u^2\le m, t\ge\frac{1}{m}\right)$$

Then, $Q_m\subset Q_{m+1}$ and $\bigcup Q_m = Q$
Furthermore, each $Q_m$ is a compact subset of $Qx(t)\in(t_0,t_1)$, $0\le t_0 < t_1 < +\infty$ is a solution of the system (3.1). Then, $x(t)$ is extendable to the point $t=t_1$ if and only if it is bounded on $(t_0,t_1)$.
Hence, a solution $y(t), t\in(t_0,T)$ is non-continuable to the right if its graph $(t,y(t)), t\in(t_0,T))$ intersects all the sets $Q_m$.
We are going to construct such a solution $y(t)$ which is continuable to the right, we may consider it defined and continuous on the interval $(t_0,t_1)$.
Now, since the graph

$G = (t, x(t)); t \in (t_0, t_1)$ is compact, there exists an integer $m$ such that $G \in Q_{m_i}$. If the number $\alpha > 0$ is sufficiently small, then for every $(a,u) \in Q$, the set

$M(a,u) = ((t,x) \in \mathbb{R}^{n+1}; |t-a| \le \alpha\, |x-u| < \alpha)$ is contained in the set

$Q_{m_1+1}$. Let $\|f\|(t,x) \le k$ on the set $Q_{m+1}$, where $k$ is a positive constant.

By Peano's theorem for every point $(a,u) \in Q_{m_1}$, there exists a solution $z(t)$ of the system (3.1) which continues $x(t)$ to the point $t_0 + qB$ and has a graph in the set $Q_{m_1+1}$ but not entirely inside the set $Q_{m_1}$. In this set $Q_{m_1+1}$, we repeat the continuation process as in the set $Q_{m_1}$. Thus, we eventually obtain a solution $y(t)$ which intersects all the sets $Q_{m_1}$, m<$m_1$ for some $m_1$ and is a continuable extension of the solution $x(t)$.

### Theorem 3.2

Let $x(t), t \in (t_0, T), 0 \le t_0 < T < +\infty$ be a non-continuable to the right solution of 3.1. Then,

$\lim_{t \to T} \to T + \|x(t)\| = +\infty$

**Proof:** Assume that, the above assertion is false. Then, there exists an increasing sequence $\{t_m\}_{m=1}^{\infty}$ such that

$t_0 \le t_m < T$, $\lim_{m\to\infty} t_m = T$ and $\lim_{m\to\infty} x(t_m)$ is such that $x(t_m) \to y$ as $m \to \infty$ with $\|y\| = L$ and $t_m$ increasing.

Let $M$ be a compact subset of $\mathbb{R}+ \times \mathbb{R}^n$ such that the point $(T, y)$ is an interior point of $m$, then we may assume that $((t_m) \times (t_m)) \in M$ for $M = 1, 2, \ldots,$

We show that for infinitely many $m$, there exists $t_m$ such that

$$t_m < T_m < t_{m+1}(t_m, x(t_m)) \in \delta M \qquad (3.2)$$

Where, $\delta M$ denotes the boundary of $M$. If this was not the case, then there would be $\varepsilon \in (0, T)$ such that $(t, x(t))$ belongs to the interior of $M$ for all $t$ with $T - \varepsilon < t < T$. Then, the condition which states that if $x(t), t \in (t_0, t_1), 0 < t_p < t_1 < +\infty$ be a solution of 3.1, then $x(t)$ is extendable to the point $t = t_1$ if and only if it is bounded on $(t_0, t_1)$ implies that the extendibility of $x(t)$ beyond $T$ is a contradiction so let

$V_\varepsilon(t,u) \le y(t, v(t,u)), (t,u) \in \mathbb{R}_+ \times \mathbb{R}^n$

$\gamma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ continuous held for a subsequence, i.e., $x(t_m)$ of positive integers satisfying

$$T_m < \overline{t_m} < t_m\left(t_m, x\left(\overline{t_m}\right)\right) \in \delta M \qquad (3.3)$$

Then, we have

$\lim_{m\to\infty}\left(\overline{t_m}\right), x\left(\overline{t_m}\right) = (T, y)$. This is a consequence of the fact that $\overline{t_m} \to T$ as $m \to \infty$ assumes and the inequality

$\left\|x\left(\overline{t_m}\right) - x(t_m)\right\| \le N\left(\overline{t_m} - t_m\right)$ where $u$ is a bound for $F$ on $M$. However, $\delta M$ is a closed set. Thus, the point $(T, y) \in M$, a contradiction to our assumption and hence the proof.

### Theorem 3.3 [On the Comparison Principle and Existence of $\mathbb{R}_+$]

Let V: $\mathbb{R}_+$ x $\mathbb{R}^n \to \mathbb{R}$ be a Lyapunov function satisfying

$$V(t,u) \le (t,u), (t,u) \in \mathbb{R}_+ \times \mathbb{R}_n \qquad (3.4)$$

and $V(t,u) \to \infty$ as $u \to \mathbb{R}$ uniformly with respect to $t$ lying in any compact set.

Here, $\gamma$: $\mathbb{R}_+ \times \mathbb{R}$ is continuous and such that for every $(t_0, u_0) \in$ $\mathbb{R}_+ \times \mathbb{R}$, the problem

$$u = \gamma(t,u) + \varepsilon, u(t_0) = u_0 + \varepsilon \qquad (3.5)$$

has a maximal solution defined on $(t_0, +\infty)$. Then, every solution of 3.1 is extendable $(t_0, +\infty)$

**Proof:** Let $(t_0, T)$ be the maximal interval of existence of solution $x(t)$ of 3.1 and assume that $T < +\infty$. Let $y(t)$ be the maximal solution of 3.1 with $y(t_0) = V(t, x(t_0))$.

Then, since $Du(t) \le \gamma(t, u(t)), t \in (t_0, +\infty) \setminus S : S$ is a continuable set… (3.6)

we have

$$V(t, x(t)) \le \gamma(t) \in (t_0, T) \qquad (3.7)$$

On the other hand, since x (t) is non-extendable to the right solution and we have

$\lim_{t \to r} \|x(t)\| = +\infty$

This implies that $V(t)$ converges to $+\infty$ as $t \to T$ but (3.7) implies that

$\limsup V(t, x(r)) \le \gamma(T)$ as $t \to T$ thus $T = +\infty$.

### Corollary 3.4

Assume that, there exists $\alpha > 0$ such that

$\|F(t,u)\| \le \gamma \in \mathbb{R}, \|u\| > \alpha$

Where, $\gamma:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ is such that every $(t_0,u_0)\in\mathbb{R}_+$ the problem (3.6) has a maximal solution defined on $(t_0,\infty)$. Then, every solution of 3.1 is extendable to $+\infty$

**Proof:** Here, it suffices to take $V(t,u)=u$ and we obtain.

$$V_\varepsilon\left(t,x(t)\right)=\lim_{h\to\infty}\left\|\frac{x(t)+hF\left(t,x(t)-x(t)\right)}{h}\right\|$$

$$\leq\left\|F(t,x(t)\right\|\leq\gamma\left(t,V\left(t,x(t)\right)\right)$$

Provided that $\|\alpha(t)\|>\alpha$. Now, let $x(t),t\in(t_0,T)$ be non-continuable to the right solution of 3.1 such that $\mathbb{R}<+\infty$. Then, for $t$ sufficiently close to $T$ from left, we have $\|x(t)\|>\alpha$.

However, if $\gamma:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ be continuous and $(t_0,x_0)\in\mathbb{R}_+\times\mathbb{R}$, $\alpha\in(o,+\infty)$ be the maximal solution of (3.6) in the interval $(t_0,t_0+\alpha)$. Let $V:(t_0,t_0+\alpha)\to\mathbb{R}$ be continuous and such that $u(t_0)\leq u_0$ and

$Du(t)\leq\gamma\left(t,u(t)\right),t\in\left(t_0,+\infty\right)\setminus S$ and $S$ is a continuable set

Then, $u(t)<S(t),t\in(t_0,t_0+\alpha)$ which implies that every solution $x(t)$ of (1.1) is extendable.

## CONCLUSION

Extension in the Real Number line is extensively shown to be real in this work by the use of the Urysohn Lemma as was seen useful in the theory of extension of ordinary differential Equations vividly expressed in Application to the theory of the Comparison Principle in Lyapunov Stability.

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