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RESEARCH ARTICLE

Approximate Solution of Time-fractional Beam Equation by Reduced Differential Transform Method

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ABSTRACT

In this paper, the reduced differential transform method (RDTM) will be applied to time-fractional order beam and beam-like equations. The RDTM produces an analytical approximate solution for the equation. An approximate analytical solution of the equation is calculated in the form of a series with few and easy computations. Three test problems are carried out to validate and illustrate the efficiency of the method. It is observed that the proposed technique is highly suitable for such problems.

Key words: Approximate analytical solutions, reduced differential transform method, time-fractional order beam equation

INTRODUCTION

Partial differential equations have numerous applications in various fields of science and engineering (Debtnath, 1997). Fractional calculus theory is a mathematical analysis tool to study the differentiation and integration to non-integer order (Oldham and Spanier, 1974). It is used in various fields of science and engineering. The fractional differential equations appear more and more frequently in different research areas and engineering applications (Podlubny, 1999; Miller and Ross, 1993). It is not always possible to find analytical solutions to these problems. In the literature, various analytical and numeric approaches have been developed for the solution partial differential equations. Since most fractional differential equations do not have exact analytic solutions, approximate and numerical techniques are used extensively.

The DTM is one of the numerical methods in solving differential equations. The concept of the DTM was first introduced by Zhou, 1986, and applied to solve initial value problems for electric circuit analysis. The method is based on Taylor's series expansion and can be applied to solve both linear and non-linear differential equations. RDTM was first introduced by Keskin *et al*., 2009; 2010a; 2010b; 2010c and 2011. This method based on the use of the traditional DTM techniques. Since RDTM has been used by many authors to obtain approximate analytical solution and in some cases exact solutions to differential equations so the investigation of exact solutions of non-linear partial differential equations (NPDEs) plays an important role in the study of non-linear physical phenomena. The investigation of exact solutions of non-linear partial differential equations (NPDEs) plays an important role in the study of non-linear physical phenomena. Many methods, exact, approximate, and purely numerical are available in literature (Sohail *et al*.; 2012a; 2012b; Cenesiz *et al*., 2010) for the solution of NPDEs. In this paper, we applied the RDTM, which is the modified version of DTM, to obtain the approximate analytical solution for time-fractional order beam and beam-like equation with appropriate initial condition. The RDTM does not require any discretization or linearization and it reduces significantly the computational work.[1-10]

Finally, the rest of the paper is organized as follows: In Section 2, we will introduce some basic definitions, theorems, and preliminary concepts of fractional calculus. The details of the RDTM formulation for time-fractional beam equation are discussed in Section 3. In Section 3.2, we apply the RDTM to solve three test examples with their solutions to show its ability and efficiency. To the end, conclusion and discussion are given in Section 4.

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FRACTIONAL CALCULUS THEORY

There are several definitions of a fractional derivative of order $\alpha>0$, in the literature due to Riemann-Liouville, Grunwald-Letnikov, Caputo, etc., (Hilfer, 2000, Miller *et al*., 1993; Oldham, 1974; Podlubny, 1999). Here, we mention the essential definition of the fractional order integral and derivative in Riemann-Liouville and Caputo sense, respectively, which are used in this work.

Definition 2.1

Let $\lambda \in R$ and $n \in N$. A real-valued function $f{:}R^+ \longrightarrow R$ belongs to C_λ if there exists $k \in R$, $k>\lambda$ and $g \in C[0,\infty)$ such that $f(x)=x^k g(x)$, for all $x \in R^+$. Moreover, $f \in \mathbb{C}_\lambda^n$ if $f^{(n)} \in C_\lambda$.

Definition 2.2

The Riemann-Liouville fractional integral of $f \in C^\lambda$ of order $\alpha \geq 0$ is defined as

$$\mathcal{T}_t^\alpha f(t)=\begin{cases} f(t) & if\ \alpha=0 \\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, & if\ \alpha>0 \end{cases}$$

Where, Γ denotes gamma function.

Definition 2.3

The fractional derivative of $f \in C_\lambda$ of the order $\alpha \geq 0$, in Caputo sense, is defined as

$$D_t^\alpha f(t)=\mathcal{T}_t^{n-\alpha} D_t^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}$$

$$\int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau$$

For $n-1<\alpha \leq n,\ n \in \mathbb{N},\ t>0,\ f \in \mathbb{C}_\lambda^n, \lambda \geq -1$. Some properties of the operator $\mathcal{T}_0^\alpha$ can be found (He, 2000). For $f \in C^\lambda$, $\lambda \geq -1$, α, $\gamma \geq 0$, $\beta \geq -1$:

$$\mathcal{T}_0^0 f(x)=f(x)$$

$$\mathcal{T}_0^\alpha t^\beta=\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}$$

$$\mathcal{T}_0^\alpha \mathcal{T}_0^\gamma f(x)=\mathcal{T}_0^{\alpha+\gamma} f(x)$$

One can see below cited references for further information on properties of fractional derivative and integral.

Lemma 2.1

Let $n-1<\alpha \leq n$, $n \in N$, and $f \in \mathbb{C}_\lambda^n, \lambda \geq -1$, then

$$D_t^\alpha \mathcal{T}_t^\alpha f(t)=f(t)$$

$$\mathcal{T}_t^\alpha D_t^\alpha f(t)=f(t)-\sum_{k=0}^n f^{(k)}(0)\frac{t^k}{k!} \text{ for } t>0.$$

ANALYSIS OF RDTM

In this section, some basic definitions and properties for RDTM are introduced as follows:

Definition 2.1

If $u(x,t)$ is analytic and continuously differentiable with respect to the space variable x and time variable t in the domain of interest, then the t-dimensional spectrum function

$$U_k(x)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=t_0} \tag{2.1}$$

is the reduced transformed function of $u(x,t)$, where α is a parameter which describes the order of time-fractional derivative. The inverse reduced differential transform of $U_k(x)$ is given by

$$u(x,t)=\sum\nolimits_{k=0}^\infty U_k(x)(t-t_0)^{k\alpha} \tag{2.2}$$

Then, combining Eqns. (2.1) and (2.2), we get:

$$u(x,t)=\sum\nolimits_{k=0}^\infty \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=t_0}(t-t_0)^{k\alpha} \tag{2.3}$$

When $t_0=0$ Eqn. (2.3) reduces to

$$u(x,t)=\sum_{k=0}^\infty \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=0} t^{k\alpha}$$

Throughout the paper, we denote the original function by $u(x,t)$.(lowercase) while it's fractional reduced differential transform by

$U_k(x)$ (*uppercase*).. From definitions, some basic properties of the Fractional reduced differential transform method are as follows:

Implementation of RDTM on time fractional beam equation

To explain how the RDTM works, we consider the general form of time-fractional order nonhomogeneous beam equation in the standard operator form

$$Lu(x,t)+Ru(x,t)=f(x,t)$$

subject to the initial conditions

$$u(x,0)=g(x)$$

Here $Lu(x,t):=D_t^{\alpha}u(x,t)$ is the fractional time derivative operator, $Ru(x,t):=\frac{\partial^4}{\partial x^4}u(x,t)$ is the linear operator and $f(x,t)$ is the nonhomogeneous source term. According to the RDTM and Table 1, we can constructing the following iteration formulas

$$U_{k+1}(x,t)=\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left[F_k(x)-RU_k(x)\right] \quad (3.1)$$

Where $R(U_k(x))$ and $F_k(x)$ are transformation of the functions $Ru(x,t)$ and $f(x,t)$, respectively, and from the initial condition, we write

Table 1: Some of the fundamental operations of the RDTM

Functional form	Transformed form
$u(x, t)$	$U_k(x)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=t_0}$
$u(x, t)\pm v(x, t)$	$U_k(x)\pm V_k(x)$
$\alpha u(x, t)$	$\alpha U_k(x)$
$x^m t^n$	$x^m\delta(k-n\delta(k)=\begin{cases}1, k=0\\0, k\neq 0\end{cases}$
$x^m t^n u(x, t)$	$x^m U_{k-n}(x)$
$u(x, t)\, v(x, t)$	$\sum_{r=0}^{k}U_r(x)V_{k-r}(x)$
$\frac{\partial^n}{\partial t^n}u(x,t)$	$(k+1)\ldots(k+n)\, U_{k+1}(x)$
$\frac{\partial}{\partial x}u(x,t)$	$\frac{\partial}{\partial x}U_k(x)$
$\frac{\partial^2}{\partial x^2}u(x,t)$	$\frac{\partial^2}{\partial x^2}U_k(x)$

$$U_0(x)=g(x) \quad (3.2)$$

Substituting Eqn. (3.2) into Eqn. (3.1), we get the values of $U_k(x)$ for all k=0,1,2,3,…. Then using the differential inverse reduced transform of $U_k(x)$, gives n term approximate solutions as

$$u^*(x,t)=\sum_{k=0}^{n}U_k(x)t^{k\alpha}$$

where n is the order of approximate solution. Therefore, the analytical approximate solutions are given by

$$u(x,t)=\lim_{n\to\infty}u_n^*(x,t)$$

Numerical applications

In this section, we used RDTM to construct analytic approximate solutions for time-fractional beam equations. For the accuracy and efficiency of the method, some illustrative examples are presented.

Example 1

Consider the time-fractional homogeneous beam equation,

$$D_t^{\alpha}u-u_{xxxx}=0 \quad (3.3)$$

Subject to initial condition

$$u(x,0)=\sin x$$

Now applying the RDTM to Eqn. (3.3) and using Table 1, we obtain the following recursive relation,

$$U_{k+1}(x)=\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left[\frac{\partial^4}{\partial x^4}U_k(x)\right] \quad (3.4)$$

And from the initial condition, we obtain

$$U_0(x)=\sin_x \quad (3.5)$$

Substituting Eqn. (3.5) into (3.4) and by straightforward iterative calculation, we get the following $U_k(x)$ values successively

$$U_1(x) = \frac{1}{\Gamma(\alpha+1)}\sin(x)$$

$$U_2(x) = \frac{1}{\Gamma(2\alpha+1)}\sin(x)$$

$$U_3(x) = \frac{1}{\Gamma(3\alpha+1)}\sin(x)$$

$$U_4(x) = \frac{1}{\Gamma(4\alpha+1)}\sin(x)$$

$$\vdots$$

Continuing with this procedure,

$$U_k(x) = \frac{1}{\Gamma(k\alpha+1)}\sin(x)$$

Then, using the differential inverse transformation (4.2), we have:

$$u(x,t) = \sin(x) + \frac{1}{\Gamma(\alpha+1)}\sin(x)t^{\alpha} + \frac{1}{\Gamma(2\alpha+1)}\sin(x)t^{2\alpha} + \dots$$

which is desired solution.

Example 2

Consider the time fractional non-homogeneous beam equation,

$$D_t^{\alpha}u - u_{xxxx} = 3x^2 \tag{3.6}$$

with initial condition

$$u(x,0)=e^x$$

Now, applying the RDTM to Eqn. (3.6) and using Table 1, we obtain the following recursive relation,

$$U_{k+1}(x) = \frac{\Gamma(\mathrm{k}\pm+1)}{\Gamma(k\alpha+\alpha+1)}\left[\frac{\partial^4}{\partial x^4}U_k(x) + 3x^2\right] \tag{3.8}$$

And from the initial condition, we obtain

$$U_0(x)=e_x \tag{3.9}$$

Substituting Eqn. (3.9) into (3.8), we get the following $U_k(x)$ values successively

$$U_1(x) = \frac{1}{\Gamma(\alpha+1)}\left[e^x + 3x^2\right]$$

$$U_2(x) = \frac{1}{\Gamma(2\alpha+1)}\left[e^x + 3x^2\right]$$

$$U_3(x) = \frac{1}{\Gamma(3\alpha+1)}\left[e^x + 3x^2\right]$$

$$U_4(x) = \frac{1}{\Gamma(4\alpha+1)}\left[e^x + 3x^2\right]$$

$$\vdots$$

Continuing with this procedure,

$$U_k(x) = \frac{1}{\Gamma(k\alpha+1)}\left[e^x + 3x^2\right]$$

Then, using the differential inverse transformation (4.2), we have:

$$u(x,t) = e^x + \frac{1}{\Gamma(\alpha+1)}\left[e^x + 3x^2\right]t^{\alpha} + \frac{1}{\Gamma(2\alpha+1)}\left[e^x + 3x^2\right]t^{2\alpha} + \dots$$

which is desired result.

Example 3

Consider the time fractional non-homogeneous beam-like equation,

$$D_t^{\alpha}u - u_x - u_{xxxx} = 0 \tag{3.10}$$

Subjected to initial condition

$$u(x,0)=e^x$$

Now, applying the RDTM to Eqn. (3.10) and using Table 1, we obtain the following recursive relation,

$$U_{k+1}(x)=\frac{\Gamma(k\pm+1)}{\Gamma(k\alpha+\alpha+1)}\left[\frac{\partial^4}{\partial x^4}U_k(x)+\frac{\partial}{\partial x}U_k(x)\right] \tag{3.11}$$

And the transformed of initial condition is

$$U_0(x)=e^x \tag{3.12}$$

Substituting Eqn. (3.12) into (3.11), we get the following $U_k(x)$ values successively

$$U_1(x)=\frac{1}{\Gamma(\alpha+1)}\left[2e^x\right]$$

$$U_2(x)=\frac{1}{\Gamma(2\alpha+1)}\left[4e^x\right]$$

$$U_3(x)=\frac{1}{\Gamma(3\alpha+1)}\left[8e^x\right]$$

$$U_4(x)=\frac{1}{\Gamma(4\alpha+1)}\left[16e^x\right]$$

$$\vdots$$

Continuing with this procedure,

$$U_k(x)=\frac{1}{\Gamma(k\alpha+1)}\left[2^{k+1}e^x\right]$$

Then, using the differential inverse transformation (4.2), we have:

$$u(x,t)=e^x+\frac{1}{\Gamma(\alpha+1)}\left[2e^x\right]t^\alpha$$
$$+\frac{1}{\Gamma(2\alpha+1)}\left[4e^x\right]t^{2\alpha}+\dots$$

which is an approximation solution.[11-15]

CONCLUSION

This paper is successfully implemented the RDTM to solve the initial value problem of time-fractional beam and beam-like equations. The fractional derivative is taken into Caputo sense. The proposed solutions are obtained in the form of power series. The validity and efficiency of RDTM have been confirmed by three test problems. The method is applied in a direct way without any linearization or discretization. Hence, this method is a powerful and an efficient technique in finding the exact solutions for wide classes of problems, also the speed of the convergence is very fast.

CONFLICTS OF INTEREST

The authors confirm that there are no conflicts of interest to declare for this publication.

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