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RESEARCH ARTICLE

# On Review of Bochner Integral and the Generalized Derivatives

C. Emmanuel Eziokwu

*Department of Mathematics, College of Physical and Applied Sciences, Michael Okpara University, Umudike, Abia State, Nigeria*

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## ABSTRACT

In this paper, we present review of integration in Banach spaces by means of definitions and theorems with special concentration on the Bochner integral. Brief touch was made on the generalized derivatives and generalized gradients (sub-differentials), and in the concluding part of this paper, we developed finite extensions of the Bochner integrals for sums and products as main results.

**Keywords:** Banach space, Hilbert space, finite measure, continuous function, integrals
**2010 Mathematics Subject Classification:** 46B25

## INTRODUCTION

In this paper, we present some important definitions and properties of spaces compromising functions on a real interval $[0,T]$ into a Banach space $X$. Such spaces and their properties are of vital importance for studying parabolic differential equations, modeling problems of plasticity, sandpile growth, superconductivity, and option pricing.

### Integration in Banach Spaces

**Definition 1.1.1:**[1] Let $(\Omega, \wp, \mu)$ be a finite measure space and $X$ a Banach space, $u: \Omega \to X$ is called strongly measurable if there exists a sequence $\{u_n\}$ of simple functions such that $\| u_n(w) - u(w) x \| \to 0$ for almost all $u$ as $n \to \alpha$.

**Definition 1.1.2 (Bochner Integral):**[2] Let $(\Omega, \wp, \mu)$ be a finite measure space and $X$ a Banach space. Then, we define the Bochner integral of simple function $u: \Omega \to X$ by

$$\int_E u du = \sum_{i=1}^{n} c_i \mu(E \cap E_1)$$

for any E $\epsilon \wp$, where $c_i$'s are fixed scales.

**Address for correspondence:**
C. Emmanuel Eziokwu
E-mail: okereemm@yahoo.com

The Bochner integral of a strongly measurable function $u$: $\Omega \to X$ is the strong limit (if it exists) of the Bochner integral of an approximating sequence $\{u_n\}$ of simple functions. That is, $\int_E u du = \lim_{n\to\infty} \int_E u_n du$.

**Remark 1.1.1**

a. The Bochner integral is independent of the approximating sequence
b. If $u$ is strongly measurable, $u$ is Bochner integrable if and only if $\| u(x) \|$ is integrable

**Definition 1.1.3:**[3] $L_p(0,T; X)$, $1 \le p < \alpha$ consists of all strongly measurable function $f$: $[0,T] \to$ X for which

$$\int_0^T \| f(t) \|_X^p dt < \infty$$

**Theorem 1.1.1:**[4] $C^m([0,T], X)$ consisting of all continuous function $f$: $[0,T] \to$ X that have continuous derivatives up to order $m$ on $[0,T]$ is a Banach space with the norm

$$\| f \| = \sum_{k=0}^{m} \sup_{0 \le t \le T} \left\| f^{(k)}(t) \right\|_X$$

**Theorem 1.1.2:**[5] $L_p$ (0,*T*; *X*) is a Banach space with the norm

$$\|f\| = \left( \int_0^T \|f(t)\|_X^P \, dt \right)^{1/2}$$

Let *X* be a Hilbert space, then $L_2$(0,*T*; *X*) is a Hilbert space with respect to inner product

$$f, gL_2(0,T;X) = \int_0^T f, g_X dt$$

**Remark 1.1.2.**

a. In $L_p$ (0,*T*; *X*), two functions are identically equal if they are equal except on a set of zero
b. $L_\alpha$ (0,*T*; *X*) denotes the space of all measurable function which are essentially bounded. It is Banach space with the norm

$$\|f\| = \sup_{0 \le t \le T} \|f(t)\|_X$$

c. If the embedding $X \subseteq Y$ is continuous, then the embedding

$$L_p(0,T:X) \subseteq L_q(0,T;Y), \quad 1 < q \le p \le \infty$$

Is also continuous.

d. Let *X** be a dual space of a Banach space *X*, then ($L_p$ (0,*T*; *X*))* the dual of $L_p$ (0,*T*; *X*) can be identified with $L_p$ (0,*T*; *X**) that is, we can write

$$\left(L_p(0,T;X)\right)^* = L_p\left(0,T;X^*\right)$$

**Definition 1.1.4 (Generalized Derivative)**[6]
Let $f \in L_1$ (0,*T*; *X*) and $g \in L_1$ (0,*T*; *X*) where *X* and *Y* are Banach spaces. The function is called the generalized derivatives of the function *f* on (0,*T*) if

$$\int_0^T \varphi^{(n)}(t) f(t) dt = (-1)^n \int_0^T \varphi(t) g(t) dt$$

$$\forall \varphi \in C_0^\infty(0,T) \qquad (1.1)$$

we write $g=f^{(n)}$.

**Remark 1.1.3**

**a)** (Uniqueness of generalized derivative). The *n-th* generalized derivative is unique. That is, if h is another *n-th* generalized derivatives, then *h*=g almost everywhere on (0,*T*) that is *h*=g in $L_1$ (0,*T*; *X*)

**b)** (Relationship between generalized derivatives and distributions). Let $f \in L_1$ (0,*T*; *X*), then a distribution *F* is associated with *f* by the relation

$$F(\varphi) = \int_0^T \varphi(t) f(t) dt \quad \forall \varphi \in C_0^\infty(0,T)$$

For each *n*, this distribution has an *nth* derivative $F^{(n)}$ defined by

$$F^{(n)}, \varphi = (-1)^n F, \varphi^{(n)} \quad \forall \varphi \in C_0^\infty(0,T)$$

If (1.1) holds, then $F^{(n)}$ can be represented by

$$F^{(n)}, \varphi = \int_0^T \varphi(t) f^{(n)}(t) dt \quad \forall \varphi \in C_0^\infty(0,T)$$

As we know, the advantage of the distribution concept is that each function $f \in L_1$ (0,*T*; *X*) possesses derivatives of every order in the distributional sense. The generalized derivative (Definition 1.1.4) singles out the cases in which by the *nth* distributional derivatives of *f* can be represented by a function $g \in L_1$ (0,*T*; *X*). In this case, we set $f^{(n)} = g$ and write briefly

$$f \in L_1(0,T;X), f^{(n)} \in L_1(0,T;Y)$$

**Theorem 1.1.3 (Generalized Derivative and Weak Convergence)**[7]

Let *X* and *Y* be Banach spaces and let the embedding $X \subseteq Y$ be continuous

Then it follows from $f_k^{(n)} = g_k$ on (0,*T*) ∀*k*, *f* fixed $n \ge 1$ and $f_k \to f$ in $L_p$ (0,*T*; *X*) as $k \to \alpha$, $g_k \to g$ in $L_p$ (0,*T*; *X*) as $k \to \alpha$, $1 \le p,q \le \alpha$ that $f^{(n)} = g$ on (0,*T*).

**Theorem 1.1.4**[7]

For a Banach space $X$, let $H^{m,p}(0,T;X)$ denote the space of all functions such that $f^m \in L_1(0,T;X)$, when $n \le m$ and $f^{(n)}$ denote the $nth$ generalized derivative of $f$. Then $H^{m,p}(0,T;X)$ is a Banach space with the norm

$$\|f\|_{H^{(m,p)}(0,T;X)} = \left(\sum_{i=0}^{m}\left\|f^{(n)}\right\|_{L_p(0,T;X)}\right)^{\frac{1}{P}} f^{(0)} = f$$

If $X$ is a Hilbert space and $p = 2$, then $H^{m,p}(0,T;X)$ is a Hilbert space with the inner product

$$f, g_{H^{m,p}(0,T;X)} = \int_0^T f^i, g_X^i\, dt$$

**Remark 1.1.4**

a. The proof of theorem (1.1.4) is similar to that of theorem which states that $H^m(\Omega)$ is Hilbert space with respect to the inner product

$$F, G = \sum_{|\alpha| \le m} D^\alpha F, D^\alpha G L_2(\Omega)$$

More generally, if $H^{m,p}(\Omega)$ denotes the space of all functions $f \in L_p(\Omega)$ such that $D^\alpha F \in L_p$, $1 \le P < \alpha$, $|\alpha| \le m$ then the space is a Banach space.

b. For $x < y$

$$\|f(y) - f(x)\|_X \le \int_x^y \|f'(t)\|_X\, dt$$

holds.

c. The embedding $H^{1,2}(0,T;X) \subset C([0,T],H)$ where $H$ is a Hilbert space, is continuous, that is, there exists a constant $k>0$such that

$$\|f\|_{C([0,T],H)} \le \|f\|_{H^{1,2}(0,T;H)}$$

## GENERAL GRADIENT (SUB-DIFFERENTIAL)

**Definition 2.1 (Lipschitz Continuity)**[7]

Let $\Omega \subset X, T$ an operator from $X$ into $Y$. We say that $T$ is Lipschitz (with modulus $\alpha \ge 0$) on $\Omega$, if $\|T(x_1) - T(x_2)\| \le \alpha\|x_1 - x_2\|$ for all $x_1, x_2 \subset \Omega$. $T$ is called Lipschitz near $x$ (with modulus $\alpha$) if, for some $\varepsilon>0$, $T$ is Lipschitz with modulus on S ($x$). If $T$ is Lipschitz near $x \in \Omega$, we say that $T$ is locally Lipschitz on $\Omega$. $\alpha$ is called the Lipschitz exponent.

**Definition 2.2 (Monotone Operators)**[7]

Let $T$: $X \to X^*$ this is called monotone if $<T_{u-}T_{v,}u-V> > 0$ for all $u,v \in X$ Note: $(\cdot,\cdot)$ denoted the duality between $X$ and $X^*$, that is, also the value $<T_{u-}T_{v,}u-V>$. In Hilbert space setting $(\cdot,\cdot)$ becomes the inner product. $T$ is called strictly monotone if $<T_{u-}T_{v,}u-V> > 0$ for all $u$, $u,v \in X$.

$T$ is called strongly monotone if there is a constant $k>0$ such that $<T_{u-}T_{v,}u-V> > k\,\|u-v\|^2$ for all $u,v \in X$.

**Definition 2.3**[7]

Let $T$: $H-2H^*$ be a multi-valued operator into $H^*$. The operator $T$ is said to be monotone if $(\xi-\eta, u-v) \ge 0$ for all $u,v \in H$ and for all $\xi \in T(u)$ and $\eta \in T$. A monotone operator $T$ is called maximal monotone if it is monotone

Proof: If $\partial F(x)$ or $\partial F(y)$ is empty, then clearly

$$\langle \partial F(x) - \partial F(y), x - y \rangle \ge 0$$

Is satisfied. If this is not the case, choose $F_1 \in \partial F(x)$ and $F_2 \in \partial F(y)$.

then

$$\left.\begin{array}{l}\langle F_1, x-y\rangle \ge F(x) - F(y) \quad \forall y \in H \\ \langle F_2, y-x\rangle \ge F(y) - F(x)\end{array}\right\}\text{Y}$$

by changing sign in (1.2.1) we get

$$\langle F_1 - F_2, x - y\rangle \ge 0$$

Hence, ů $\partial F(x)$ is a monotone operator

## MAIN RESULT

**Theorem 3.1**

Given $(\Omega, \wp, \mu)$ a finite measure such that

$$\int_E u\, d\mu = \sum_{i=1}^{n} c_i \mu(E \cap E_1)$$

Exist then for all $j$ satisfying $1 \le j \le m < \alpha$, the following holds

i. The sum $\sum_j \int_E u\,d\mu = j\sum_{i=1}^{n} c_i\mu(E\cap E_i)$

ii. The product

$$\prod\left(\sum\int_E u\,d\mu\right) = \left(\sum_{i=1}^{n} c_i\mu(E\cap E_i)\right)^j$$

***Proof***

1. By induction: If

$j=1, \sum_1 \int_E u\,d\mu = 2\sum_{i=1}^{2} c_i\mu(E\cap E_i)$ which is the celebrated Bochner integral. If $j$=2,

$$\sum_{i=1}^{2}\int_E u\,d\mu = 2\sum_{i=1}^{2} c_i\mu(E\cap E_i)$$

Assume it is true for $n=k$ we now prove it is true for $n=k+1$. Hence

$$\sum_{k+1}\int_E u\,d\mu = \sum_k\int_E u\,d\mu + \int_E u\,d\mu$$

$$= k\sum_{i=1}^{n} c_i\mu(E\cap E_i) + \sum_{i=1} c_i\mu(E\cap E_i)$$

$$= (k+1)\sum_{i=1}^{n} c_i\mu(E\cap E_i)$$

Therefore, since this is true for $n=k+1$, hence it is true for $n=k$ and then the proof that

$$\sum_j\int_E u\,d\mu = j\sum_{i=1}^{n} c_i\mu(E\cap E_i)$$

2. For $j$=1, the claim is obviously true and for $n$=2

$$\sum_2\int_E u\,d\mu = \prod_2\left(\sum_{i=1}^{n}\int_E u\,d\mu\right) = \left(\sum_{i=1}^{n} c_i\mu(E\cap E_i)\right)^2$$

Assume it is true for $n=k$, then, we now prove it is true for $n=k+1$ such that

$$\left(\sum_{i=1}^{k+1}\int_E u\,d\mu\right) = \left(\sum_{i=1}^{k+1} c_i\mu(E\cap E_i)\right)$$

$$= \left(\sum_{i=1}^{k} c_i\mu(E\cap E_i)\right)^k \left(\sum_{i=1}^{n} c_i\mu(E\cap E_i)\right)^1$$

$$= \left(\sum_{i=1}^{n} c_i\mu(E\cap E_i)\right)^{k+1}$$

Since it holds for $n=k+1$, hence, it holds for $n=k$. Therefore

$$\left(\sum_{i=1}^{k+1}\int_E u\,d\mu\right) = \left(\sum_{i=1}^{n} c_i\mu(E\cap E_i)\right)^j$$

A good example of this main result can be seen in the example on integration by part stated below.

**Example 3.1:**

Let $f(x) = |x|, f: R \to R$

$$\partial f(x) = \{\operatorname{sgn} x\} \text{if } x \ne 0, \quad \operatorname{sgn} x = \frac{x}{|x|}, \quad x \ne 0$$

$$\partial f(x) = [-1,1] \quad \text{if } x = 0$$

More examples on generalized gradient can be generated using the ones on Quatrata, Krevara, Zowe and Rockafella and Wets.

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