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RESEARCH ARTICLE

Limiting Ratios of Generalized Recurrence Relations

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ABSTRACT

Recurrence relations are used to study the behavior of terms of a particular sequence. In this paper, I am primarily focused on determining the ratio of $(n + 1)^{\text{th}}$ term to n^{th} term which is called as limiting ratio. Beginning with the recurrence relations for Fibonacci sequence, I extended it gradually to obtain a more generalized recurrence relation and determine the limiting ratio for that general case. Several figures are provided to verify and enhance the understanding of the limiting results obtained.

Key words: Recurrence relation, Fibonacci sequence, generalized recurrence relation, characteristic equation, limiting ratio

INTRODUCTION

Ever since, Italian mathematician and merchant Leonardo Fibonacci published his seminal book "Liber Abaci" in 1202 CE, the concept of Fibonacci sequence continues to be known widespread to mathematical community throughout the globe. The terms of the Fibonacci sequence can be generated through nice and simple recurrence relation. Several generalizations of Fibonacci sequence and its associated Lucas sequence were studied extensively by many mathematicians and the significant results regarding these concepts were published in the exclusive journal Fibonacci Quarterly, a journal devoted for publishing results focusing on Fibonacci sequence and its related ideas. In this paper, we try to generalize the recurrence relation used for Fibonacci sequence and derive new results with respect to the limiting ratios. Several figures were provided to illustrate and verify the results obtained in this paper. The final result will provide a new insight in understanding the behavior of limiting ratio of generalized recurrence relations.

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DEFINITIONS

The recurrence relation of Fibonacci sequence is given by
$P(n+2)=P(n+1)+P(n)$, $n\geq 0$, $P(0)=1$, $P(1)=1$ (2.1). In (2.1), we observe that except the first two terms, each term of the sequence is sum of the two previous terms. The generalized recurrence relation of Fibonacci type sequence is defined by

$P(n+m)=P(n+m-1)+P(n+m-2)+\ldots+P(n+1)+P(n)$, $n\geq 0$, $m\geq 2$ (2.2)

where $P(0)=P(1)=P(2)=\ldots=P(m-2)=P(m-1)=1$. In (2.2), we observe that except for the first m terms, each term is sum of the previous m terms of the sequence.
The ratio of $(n+1)$\th term to the nth term of a sequence as $n\to\infty$ is defined as the limiting ratio of the sequence. We denote the limiting ratio by λ.

Thus, $\lambda=\dfrac{P(n+1)}{P(n)}$ as $n\to\infty$ (2.3)

If λ is the limiting ratio, then for any integer $r\geq 1$ and as $n\to\infty$ we have

$$\frac{P(n+r)}{P(n)}=\frac{P(n+r)}{P(n+r-1)}\times\frac{P(n+r-1)}{P(n+r-2)}\times\cdots\times\frac{P(n+2)}{P(n+1)}\times\frac{P(n+1)}{P(n)}=\lambda\times\lambda\times\cdots\times\lambda\times\lambda=\lambda^r \quad (2.4)$$

SPECIAL CASES

When $m = 2$

If $m = 2$, then according to (2.1), the recurrence relation would be

$P(n+2)=P(n+1)+P(n)$, $n\geq 0$, $P(0)=1$, $P(1)=1$ (3.1). We notice that this case provides the Fibonacci sequence. The characteristic equation of this recurrence relation is given by $x^2-x-1=0$ (3.2). The roots of equation (3.2) being quadratic equation, are given by $x=\frac{1\pm\sqrt{5}}{2}$. Among these, the positive root is given by $x=\frac{1+\sqrt{5}}{2}$ (Figure 1). This number which is approximately 1.618 is called the Golden Ratio.

Now from (3.1), we have $\frac{P(n+2)}{P(n)}=\frac{P(n+1)}{P(n)}+1$.

If λ is the limiting ratio of (3.1) then as $n\to\infty$ from (2.4), we get $\lambda^2-\lambda-1=0$. But this is precisely the same equation as (3.2). Beginning with 1, since each term of the Fibonacci sequence is non-decreasing, the limiting ratio should be positive. Hence, the limiting ratio λ is the positive real root of $\lambda^2-\lambda-1=0$ which is $\lambda=\frac{1+\sqrt{5}}{2}$. This number is called the Golden Ratio. Thus, if $m = 2$, then the limiting ratio (of the Fibonacci sequence) is the Golden Ratio given by 1.618 approximately (3.3).

When $m = 3$

If $m = 3$, then from (2.2), we get

$P(n+3)=P(n+2)+P(n+1)+P(n)$, $n\geq 0$, $P(0)=1$, $P(1)=1$, $P(2)=1$ (3.4). The characteristic equation of (3.4) is given by $x^3-x^2-x-1=0$ (3.5). By Newton – Raphson method, we see that the positive real root of the polynomial in (3.5) is 1.83928 approximately. Figure 2 verifies this fact.

From (3.4), we get $\frac{P(n+3)}{P(n)}=\frac{P(n+2)}{P(n)}+\frac{P(n+1)}{P(n)}+1$.

If λ is the limiting ratio of (3.4) then as $n\to\infty$ from (2.4), we get $\lambda^3-\lambda^2-\lambda=0$. But this is precisely the same equation as (3.5). Hence, the limiting ratio of (3.4) is the positive real root of (3.5) which is 1.83928 approximately (3.6).

When $m = 4$

If $m = 4$, then from (2.2), we get

$P(n+4)=P(n+3)+P(n+2)+P(n+1)+P(n)$, $n\geq 0$, $P(0)=1$, $P(1)=1$, $P(2)=1$, $P(3)=1$ (3.7). The characteristic equation of (3.7) is given by $x^4-x^3-x^2-x-1=0$ (3.8). By Newton – Raphson method, we see that the positive real root of the polynomial in (3.8) is 1.92756 approximately. Figure 3 verifies this fact.

From (3.7), we get

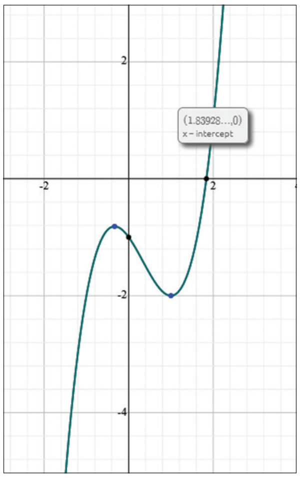

Figure 2: Graph of $y=x^3-x^2-x-1$

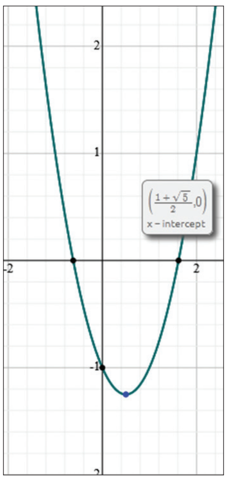

Figure 1: Graph of $y = x^2-x-1$

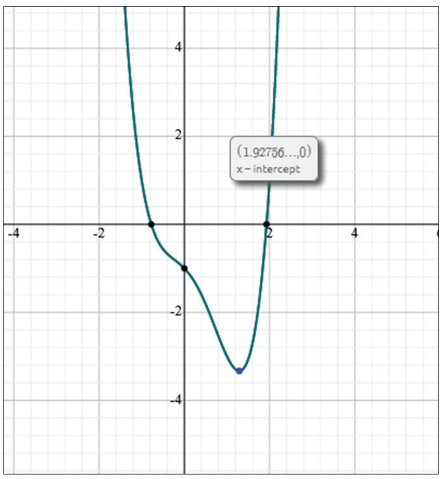

Figure 3: Graph of $y=x^4-x^3-x^2-x-1$

$\frac{P(n+4)}{P(n)}=\frac{P(n+3)}{P(n)}+\frac{P(n+2)}{P(n)}+\frac{P(n+1)}{P(n)}+1$. If λ is the limiting ratio of (3.7) then as $n\to\infty$ from (2.4), we get $\lambda^4-\lambda^3-\lambda^2-\lambda-1=0$. But this is precisely the same equation as (3.8). Hence, the limiting ratio of (3.7) is the positive real root of (3.8) which is 1.92756 approximately (3.9).

When $m = 5$

If $m = 5$, then from (2.2), we get
$P(n+5)=P(n+4)+P(n+3)+P(n+2)+P(n+1)+P(n)$, $n\geq 0$
$P(0)=1, P(1)=1, P(2)=1, P(3)=1, P(4)=1$ (3.10)
The characteristic equation of (3.10) is given by $x^5-x^4-x^3-x^2-x-1=0$ (3.11). By Newton – Raphson method, we see that the positive real root of the polynomial in (3.11) is 1.96594 approximately. Figure 4 verifies this fact.
From (3.10), we get $\frac{P(n+5)}{P(n)}=\frac{P(n+4)}{P(n)}+\frac{P(n+3)}{P(n)}+\frac{P(n+2)}{P(n)}+\frac{P(n+1)}{P(n)}+1$.
If λ is the limiting ratio of (3.10), then as $n\to\infty$ from (2.4), we get $\lambda^5-\lambda^4-\lambda^3-\lambda-1=0$. But this is precisely the same equation as (3.11). Hence, the limiting ratio of (3.10) is the positive real root of (3.11) which is 1.96594 approximately (3.12).

When $m = 10$

If $m = 10$, then from (2.2), we get
$P(n+10)=P(n+9)+P(n+8)+P(n+7)+\ldots+P(n+1)+P(n)$, $n\geq 0$,
$P(0)=P(1)=\ldots=P(9)=1$ (3.13)
The characteristic equation of (3.13) is given by $x^{10}-x^9-x^8-x^7-x^6-x^5-x^4-x^3-x^2-x-1=0$ (3.14). By Newton – Raphson method, we see that the positive real root of the polynomial in (3.14) is 1.99901 approximately. Figure 5 verifies this fact.

From (3.13), we get $\frac{P(n+10)}{P(n)}=\frac{P(n+9)}{P(n)}+\frac{P(n+8)}{P(n)}+\cdots+\frac{P(n+2)}{P(n)}+\frac{P(n+1)}{P(n)}+1$.

If λ is the limiting ratio of (3.13), then as $n\to\infty$ from (2.4), we get
$\lambda^{10}-\lambda^9-\lambda^8-\lambda^7-\lambda^6-\lambda^5-\lambda^4-\lambda^3-\lambda^2-\lambda-1=0$.
But this is precisely the same equation as (3.14). Hence, the limiting ratio of (3.13) is the positive real root of (3.14) which is 1.99901 approximately (3.15).
Thus, through the five cases for m = 2, 3, 4, 5, and 10 in sections 3.1 to 3.5, respectively, we noticed that as m increases, the limiting ratios of the generalized recurrence relation defined in (2.2) approaches 2. I formally prove this through the following theorem.

THEOREM 1

The limiting ratio of the generalized recurrence relation converges to 2
Proof: The generalized recurrence relation (as defined in (2.2)) is given by
$P(n+m)=P(n+m-1)+P(n+m-2)+\ldots+P(n+1)+P(n)$, $n\geq 0$, $m\geq 2$ (4.1)
where $P(0)=P(1)=P(2)=\ldots=P(m-2)=P(m-1)=1$
The characteristic equation of the generalized recurrence relation is given by $x^m-x^{m-1}-x^{m-2}-\ldots-x^3-x^2-x-1=0$ (4.2)

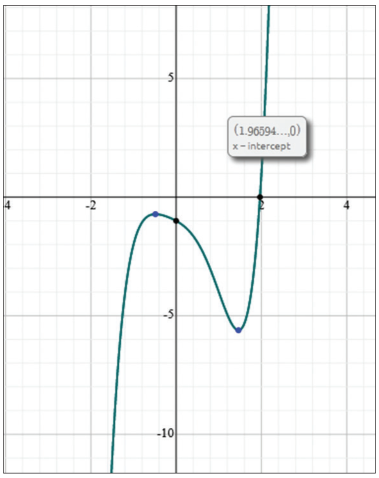

Figure 4: Graph of $y=x^5-x^4-x^3-x^2-x-1$

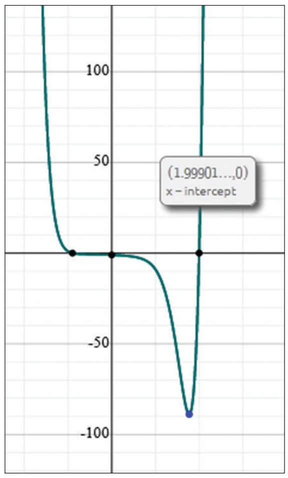

Figure 5: Graph of $y=x^{10}-x^9-x^8-x^7-x^6-x^5-x^4-x^3-x^2-x-1$

From (4.1), we have

$$\frac{P(n+10)}{P(n)}=\frac{P(n+9)}{P(n)}+\frac{P(n+8)}{P(n)}+\cdots+\frac{P(n+2)}{P(n)}+\frac{P(n+1)}{P(n)}+1 \quad (4.3)$$

Now using (2.4), as $n\to\infty$ (4.3) can be written as

$$\lambda^m-\lambda^{m-1}-\lambda^{m-2}-\ldots-\lambda^3-\lambda^2-\lambda-1=0 \quad (4.4)$$

We notice that (4.4) is the same equation as the characteristic equation of the generalized recurrence relation given by (4.2). Hence, the positive real root of the characteristic equation will be limiting ratio of the generalized recurrence relation (4.1).

If $f(\lambda)=\lambda^m-\lambda^{m-1}+\lambda^{m-2}-\ldots-\lambda^3-\lambda^2-\lambda-1$, then we find that $f(1)=-(m-1)<0$ since $m\geq 2$. Similarly, $f(2)=2^m-2^{m-1}-2^{m-2}-\ldots-2^3-2^2-2-1=2^m-(2^m-1)=1>0$.

Hence, the positive real root λ of (4.4) must lie between 1 and 2 for all $m\geq 2$.

Since, $(\lambda-1)(\lambda^{m-1}+\lambda^{m-2}+\ldots+\lambda^3+\lambda^2+\lambda+1)=\lambda^m-1$ (4.5), using (4.4) in (4.5), we get $(\lambda-1)\lambda^m=1$. Simplifying this equation, we get $1+\lambda^{m+1}=2\lambda^m$ giving $\lambda+\frac{1}{\lambda^m}=2$ (4.6).

Since $\lambda>1$, $\frac{1}{\lambda^m}\to 0$ as $m\to\infty$. Hence, from (4.6), we see that $\lambda\to 2$ as $m\to\infty$.

Thus, the limiting ratio λ of the generalized recurrence relation converges to 2.

This completes the proof.

To know more about generalized recurrence relations and their behavior see.[1-7]

CONCLUSION

Generalizing the recurrence relation of the Fibonacci sequence, we defined new recurrence relation in (2.2). In the case if $m = 2$, we see that it reduces to Fibonacci sequence. It is well known that the limiting ratio of the Fibonacci sequence is one of the most famous real number called Golden Ratio given by 1.618 approximately. We proved this fact in this paper in the section 3.1. Considering higher values of m as 3, 4, 5, and 10 in sections 3.2, 3.3, 3.4, and 3.5, we obtained limiting ratios of each case. We also drew graphs of the characteristic equations of all recurrence relations in sections 3.1–3.5 to verify the obtained limiting ratio values. The roots were determined through Newton – Raphson method and graphs were constructed using Desmos Graphing Calculator online software. In analyzing the limiting ratio values obtained in section 3, we noticed that as we increase the value of m, the limiting ratios were as close as possible to 2. This fact was indeed proved to be true through theorem 1 of section 4. Thus, for each integer value of $m\geq 2$, the limiting ratios approach the number 2. This result is the main objective of this paper. We can extend the generalized recurrence relations in several ways to obtain further results on limiting ratios and discuss their convergence as we did in theorem 1 of this paper.

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