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On the Generalized Real Convex Banach Spaces

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ABSTRACT

Given a normed linear space X, suppose its second dual X^{**} exists so that a canonical injection J: $X+X^{**}$ also exists and is defined for each $x \in X$ by $J(x) = Ø_x$ where $Ø_x$: $X^{**} \to \mathbb{R}$ is given by $Ø_x(f)=<f, x>$ for each $f \in X^*$ and $<J(x),(f)>=<f, x>$ for each $f \in X^{**}$. Then, the mapping J is said to be embedded in X^{**} and X is a reflexive Banach space in which the canonical embedding is onto. In this work, a general review of Kakutan's, Helly's, Goldstein's theorem, and other propositions on the convex spaces was X-rayed before comprehensive results on uniformly convex spaces were studied, while the generalization of these results was discussed in section there as main result along the accompanying proofs to the result.

Key words: Banach spaces and the duals canonical injection, Canonical embedding, Isometry
Weak and weak star topologies, Reflexive Banach space, Convex Banach spaces uniformly convex functions.

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REFLEXIVESPACES

A known fact is that for any normed linear space X, the space X^* of all bounded linear functionals on X is a Banach space, and as a Banach space X^* has its own dual space which we denote by $(X^*)^*$ or simply by X^{**} which is often referred to as the second dual space of X. Hence, there exists a canonical injection J: $X+X^{**}$ of X into X^{**} defined for each $x \in X$ by

$J(x) = Ø_x$

where $Ø_x$: $X^{**} \to \mathbb{R}$ is given by

$Ø_x(f)=<f, x>$

for each $f \in X^*$. Thus, $<J(x),(f)>=<f, x>$ for each $f \in X$.

Note the following facts:

i. J is linear.

ii. $\|J_x\|=\|x\|$ for all $x \in X$ meaning that J is Isometry.

In general, the map J needs not be onto, and consequently, we always identify X as a subspace of X^{**}. Since an isometry is always a one-to-one map, it follows that J is an isomorphism onto $J(X) \subset X^{**}$. The mapping J defined above is called the canonical map of X into X^{**} and the space X is said to be embedded in X^{**}, hence the following definitions.

Definition 1.1[1]: Let X is a normed linear space and let J is the canonical embedding of X into X^{**}. If J is onto, then X is called reflexive, that is, a reflexive Banach space is one in which the canonical embedding is onto.

Proposition 1.1[2]: Let X is a finite dimensional normed linear space. Then, the strong, weak and weak star topologies coincide.

Theorem 1.1: (Kakutan's theorem)[3]

Let X is a Banach space. Then, X is reflexive if and only if

$$B_x = \{x \in X : \|x\| \leq 1\}$$

is weakly compact.

Lemma 1.1: (Helly's theorem)[4]

Let X is a Banach space, $\{f_i\}_{i=1}^{\infty} \in X^*$ and $\{\alpha_i\}_{i=1}^{\infty} \in \mathbb{R}$ fixed. Then, the following properties are equivalent

i. $\forall \varepsilon > 0, \ \exists x \in X \ni \|x\| \leq 1$ and

$$\|\langle f_n, x_n \rangle\| < \varepsilon, \ \forall i = 1, 2, \ldots, n$$

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ii. $$\left\|\sum_{i=1}^{n}\beta_i\alpha_i\right\| \le \left\|\sum_{i=1}^{n}\beta_i f_i\right\| \ \forall \beta_1,\beta_2,...,\beta_n \in \mathbb{R}$$

Lemma 1.2: (Goldstein's theorem)[5]
Let X is a Banach space then $J(B_x)$ is dense in $(B_{x^{**}}, w)$.
Lemma 1.3: Let X and Y are Banach spaces and let $T(X,S) \to (Y,S)$ is a linear continuous map. Then, $T(X,w) \to (Y,w)$
is continuous and conversely.
Proposition 1.2[6]: Let X is a reflexive Banach space and let K a closed subspace of X, then K is reflexive.
Corollary 1.1[7]: Let X is a Banach space, then X is reflexive if and only if X^* is.
Corollary 1.2: Let X is a Banach space, K a closed bounded convex non-empty subset of X. Then, K is weakly compact.

UNIFORMLY CONVEX BANACH SPACES

Definition 2.1[8]: A subset M of a vector space X is said to be convex if $x_1, x_2 \in M$. Implies that the set $W=\{v=ax_1+(1-a)x_2 : 0\le\alpha\le 1\}$ is a subset of M. The set W is called a closed segment, while x_1 and x_2 are called the boundary points of the segment W and any other point of W is called the interior point of W.
Definition 2.2 (Strict Convexity)[9]
A strict convex norm is a norm such that for all x_1, x_2 of norm 1,

$$\|x_1 + x_2\| < 2$$

A normed space with such a norm is called a strictly convex normed space.
Definition 2.3 (Best Approximation)[9]
Let $X=(X,\|\bullet\|)$ is a normed space and suppose that any given $x_1 \in X$ is to be approximated by a $x_2 \in Y$ where Y is a fixed subspace of X. We let δ denote the distance from x_1 to Y. By definition
$$\delta = \delta(x_1, Y) = \inf_{y\in Y} \|x_i - x_2\|$$

Clearly δ depends on both x_1 and Y which we keep fixed so that the simple notation δ is in order if there exists a $x_0 \in Y$ such that
$$\|x - x_0\| = \delta$$

then y_0 is called a best approximation to x_1 out of Y.
Lemma 2.1 (Convexity)[9]
In a normed space $X=(X,\|\bullet\|)$, the set M of best approximations to a given x_1 out of a subspace Y of X is convex.

Lemma 2.2 (Strict Convexity)[9]
We have that
(a) The Hilbert space is strictly convex
(b) The space $C[a,b]$ is not strictly convex
Definition 2.4 (Uniformly Convex Banach Spaces)[10]
Let X is a Banach space, and $S_r(x_0)$, $B_r(x_0)$ denote the sphere and the open ball, respectively, centered at x_0 and with radius $r>0$.

$$S_r(x_0) = \{x \in X : \|x - x_0\| = r\}$$
$$B_r(x_0) = \{x \in X : \|x - x_0\| < r\}$$

A Banach space X is called uniformly convex if for any $\varepsilon \in (0,2]$ there exists a $\delta=\delta(\varepsilon)>0$ such that if $x_1, x_2 \in \varepsilon$ with $\|x_1\|\le 1$, $\|x_2\|\le 1$ and $\|x_1-x_2\|\le\varepsilon$, then

$$\left\|\frac{1}{2}(x_1 + x_2)\right\| \le 1 - \delta$$

Definition 2.5[10]
A normed space is called strictly convex if for all $x_1, x_2 \in X$, $x_1 \ne x_2$ $\|x_1\|=\|x_2\|=1$, we have

$$\|\lambda x_1 + (1-\lambda)x_2\| < 1 \ \forall \lambda \in (0,1)$$

Theorem 2.1.
Every inner product space H is uniformly convex.
Example 2.1: $X=L_p$ spaces $1 < p < \infty$ are uniformly convex.
Example 2.2: ℓ_p $(1 < p < \infty)$ is uniformly convex
Example 2.3: If $X=\ell_p$, then for $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$ and for each pair $x,y \in X$, the following inequalities hold
i.

$$\left\|\frac{1}{2}(x+y)\right\|^q + \left\|\frac{1}{2}(x-y)\right\|^q \le \left\|2^{p-1}\left(\|x\|^p + \|y\|^p\right)\right\|^{q-1},$$
$1 \le p \le 2$

ii.

$$\|x+y\|^p + \|x+y\|^q \le 2^{p-1}\left(\|x\|^p - \|y\|^p\right), \ 2 \le p \le \infty$$

Example 2.4: The spaces ℓ_1 and ℓ_∞ are not uniformly convex as well as the space $C[a,b]$ of all real valued continuous functions on the compact interval $[a,b]$ endowed with the "sup norm."

Proposition 2.1[11]:
Let X is a uniformly convex Banach space, then for any δ >0, ε >0 and arbitrary vectors x_1, $x_2 \in X$ with $\|x_1\| \leq \delta$, $\|x_2\| \leq \delta$ and $\|x_1 - x_2\| \leq \varepsilon$ there exists a δ >0 such that

$$\left\|\frac{1}{2}(x_1 + x_2)\right\| \leq \left[1 - \delta\left(\frac{\varepsilon}{\delta}\right)\right]\delta$$

Proposition 2.2[11]:
Let X is a uniformly convex Banach space and let $\alpha \in (0,1)$ and ε >0. Then, for any δ >0, if x_1, $x_2 \in X$ are such that $\|x_1\| \leq \delta$, $\|x_2\| \leq \delta$ and $\|x_1 - x_2\| \leq \varepsilon$, then there exists $\delta = \delta\left(\frac{\varepsilon}{\delta}\right) > 0$ such that

$$\|\alpha x_1 + (1-\alpha)x_2\| \leq \left[1 - 2\delta\left(\frac{\varepsilon}{\delta}\right)\min\{\alpha, 1-\alpha\}\right]^{\delta}$$

Theorem 2.2[11]:
Every uniformly convex space is strictly convex.

Definition 2.6 (Convex Function)[12]
A function $f: \mathbb{R}^n \to R$ is said to be convex if its domain $D(f)$ is a convex set and for every x_1, $x_2 \in D(f)$,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

where $0 \leq \lambda \leq 1$.
Lemma 2.3[12]: Every convex function f with convex domain in $\mathbb{R}$ is continuous.
Definition 2.7[12]:
Let X is a normed space with dim $X \geq 2$. The modulus of convexity of X is the function

$$\delta_x : (0,2] \to [0,1]$$

defined by

$$\delta_x(\varepsilon) = \inf\left\{1 - \left\|\frac{x_1 + x_2}{2}\right\| : \|x_1\| = \|x_2\| = 1; \; \varepsilon = \|x_1 - x_2\|\right\}$$

where in particular for an inner product H, we have

$$\delta_H(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$$

Lemma 2.4[12]: Let X is a normed space with dim $X \geq 2$. Then,

$$\delta_x(\varepsilon) = \inf\left\{1 - \left\|\frac{x_1 + x_2}{2}\right\| : \|x_1\| \leq 1; \|x_2\| \leq 1; \; \varepsilon \leq \|x_1 - x_2\|\right\}$$

$$= \inf\left\{1 - \left\|\frac{x_1 + x_2}{2}\right\| : \|x_1\| \leq 1; \|x_2\| \leq 1; \; \varepsilon = \|x_1 - x_2\|\right\}$$

This lemma implies $\delta_x(0)=0$.
Lemma 2.5[12]: For every normed space X, the function $\delta_x(\varepsilon)/\varepsilon$ is decreasing on (0,2].
Theorem 2.3[12]: The modulus of convexity of a normed space X, δ_x is a convex and continuous function.
Theorem 2.4[12]: A normed space X is uniformly convex spaces of $\delta_x(\varepsilon)>0$ for all $\varepsilon \in (0,2]$.
Theorem 2.5[12]: If X is an arbitrary uniformly convex space, then

$$\delta_x(\varepsilon) \leq 1 - \sqrt{\frac{\varepsilon^2}{4}}$$

Theorem 2.6: (Milman Pettis theorem) [12]
If X is a uniformly convex Banach space, then X is reflexive.

MAIN RESULTS ON GENERALIZED CONVEX SPACES

Definition 3.1: (Generalized Convex Space)
From definition 2.1 above, a union of subsets $\bigcup_{i=1}^{n} M_i$ of vector space X is said to be convex if $\bigcup_{i=1}^{n} y_i \bigcup_{i=1}^{n} z_i \in \bigcup_{i-1}^{n} M_i$ implies that the set

$$\bigcup_{i=1}^{n} W_i = \left\{\bigcup_{i=1}^{n} v_i = \alpha\bigcup_{i=1}^{n} y_i + (1-\alpha)\bigcup_{i=1}^{n} z_i; \; 0 \leq \alpha \leq 1\right\}$$

is a subset of $\bigcup_{i=1}^{n} M_i$. The set $\bigcup_{i=1}^{n} W_i$ is called a closed segment while $\bigcup_{i=1}^{n} y_i$ and $\bigcup_{i=1}^{n} z_i$ are called the boundary sets of segment $\bigcup_{i=1}^{n} W_i$ and any other point set of $\bigcup_{i=1}^{n} W_i$ is called the interior point set of $\bigcup_{i=1}^{n} W_i$.

Definition 3.2: (Generalized strict convexity)
A generalized strict convex norm is a norm such that for all $\bigcup_{i=1}^{n} x_i$, $\bigcup_{i=1}^{n} y_i$ of norm 1,

$$\left\|\bigcup_{i=1}^{n} x_i + \bigcup_{i=1}^{n} y_i\right\| = \left\|\bigcup_{i=1}^{n}(x_i + y_i)\right\| = \bigcup_{i=1}^{n}\|x_i + y_i\| < 2$$

such a normed space is called a strictly generalized normed space.

Definition 3.3: (Generalized Best Approximation)

Let $X=(X,\|\cdot\|)$ is a normed space and suppose that any given set $\bigcup_{i=1}^{n} x_i \in X$ is to be approximated by a $\bigcup_{i=1}^{n} y_i \in Y$ where Y is a fixed subspace of X. We let δ denote the distance from $\bigcup_{i=1}^{n} x_i$ to Y. By definition

$$\delta = \delta\left(\bigcup x_i, y\right) = \inf_{y_i \in Y} \bigcup_{i=1}^{n}\|x_i - y_i\|$$

Clearly, δ depends on both $\bigcup_{i=1}^{n} x_i$ and Y which we keep fixed so that the simple notation δ is in order. If there exists a $y_0 \in Y$ such that

$$\left\|\bigcup_{i=1}^{n} x_i - y_0\right\| = \bigcup_{i=1}^{n}\|x_i - y_0\| = \delta$$

Then, y_0 is called a best approximation to $\bigcup_{i=1}^{n} x_i$ out of Y.

Lemma 3.1: (Generalized Convexity)

In a normed space $X=(X,\|\cdot\|)$, the generalized set M of best approximations to a given $\bigcup_{i=1}^{n} x_1$ out of a subspace Y of X is convex.

Lemma 3.2: (Generalized Strict Convexity)

a. The Hilbert space is strictly generally convex
b. The space $C[a,b]$ is not strictly generally convex

Definition 3.4 (Uniformly Convex Banach Spaces)

Given an arbitrary Banach space X, for $x_0 \in X$ and let $\bigcup_{i=1}^{n} S_{r_i}(x_0)$ be the sphere centered at x_0 with radius $\bigcup_{i=1}^{n} r_i > 0$ such that

$$\bigcup_{i=1}^{n} S_{r_i}(x_0) = \left\{\bigcup_{i=1}^{n} x_i \in X : \bigcup_{i=1}^{n}\|x_i - x_0\| = \bigcup_{i=1}^{n} r_i\right\}$$

Then, a normed space X is called generalized uniformly convex if for any (0,2] there exists a $\delta = \bigcup_{i=1}^{n}\delta_i(\varepsilon_i)$ such that if $\bigcup x_i, \bigcup x_j \in X$ with $\bigcup_{i=1}^{n}\|x_i\| = 1, \bigcup_{j=1}^{n}\|x_j\| = 1$ and $\bigcup_{\substack{i=1\\j=1}}^{n}\|x_i - x_j\| \geq \bigcup_{i=1}^{n}\varepsilon_i$ then

$$\bigcup_{\substack{i=1\\j=1}}^{n}\left\|\frac{1}{2}(x_i + x_j)\right\| \leq \left(1 - \bigcup \delta_i\right)$$

We also note the following useful definition.

A generalized normed space X is uniformly convex if for any $\bigcup_{i=1}^{n}\varepsilon_i \in (0,2]$, there exists $\bigcup_{i=1}^{n}\delta_i = \bigcup_{i=1}^{n}\delta_i(\varepsilon_i) > 0$ such that if $\bigcup_{i=1}^{n} x_i, \bigcup_{i=1}^{n} x_j \in X$ with $\bigcup_{i=1}^{n}\|x_i\| \leq 1,\ \bigcup_{j=1}^{n}\|x_j\| \leq 1$ and $\bigcup_{\substack{i=1\\j=1}}\|x_i - x_j\| \geq \bigcup \varepsilon_i$ then

$$\bigcup_{\substack{i=1\\j=1}}^{n}\left\|\frac{1}{2}(x_i + x_j)\right\| \leq \bigcup_{i=1}^{n}(1 - \delta_i)$$

Definition 3.5

A normed space is called strictly convex in the generalized sense if for all $\bigcup_{i=1}^{n} x_i, \bigcup_{j=1}^{n} x_j \in X,\ x_i \neq x_j$ $\bigcup_{i=1}^{n}\|x_i\| = 1, \bigcup_{j=1}^{n}\|x_j\| = 1$, we have

$$\bigcup_{\substack{i=1\\j=1}}^{n}\|\lambda x_i + (1-\lambda)x_j\| < 1 \quad \forall \lambda \in (0,1)$$

Theorem 3.1.

Every inner product space H is uniformly convex in the generalized sense.

Example 3.1: $X=L_p$ spaces $1< p< \infty$ are uniformly convex in the generalized sense.

Example 3.2: The ℓ_p ($1< p< \infty$) is uniformly convex in the generalized sense.

Example 3.3: If $X=\ell_p$, then for $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$ and for each pair $\bigcup_{i=1}^{n} x_i, \bigcup_{j=1}^{n} x_j \in X$, the following inequalities hold.

i.

$$\bigcup_{\substack{i=1\\j=1}}^{n}\left\|\frac{1}{2}(x_i + x_j)\right\|^q + \bigcup_{\substack{i=1\\j=1}}^{n}\left\|\frac{1}{2}(x_i - x_j)\right\|^q$$

$$\leq \bigcup_{\substack{i=1\\j=1}}^{n}\left\|2^{p-1}\left(\|x_i\|^p + \|x_j\|^p\right)\right\|^{q-1}, 1 \leq p \leq 2$$

and

ii. $$\bigcup_{\substack{i=1\\j=1}}^{n}\|x_i+x_j\|^p+\bigcup_{\substack{i=1\\j=1}}^{n}\|x_i+x_j\|^q \le 2^{p-1}\left(\bigcup_{\substack{i=1\\j=1}}^{n}\|x_i\|^p-\bigcup_{\substack{i=1\\j=1}}^{n}\|x_j\|^p\right),\ 2\le p\le\infty$$

Example 3.4: The spaces ℓ_1 and ℓ_∞ are not uniformly convex in the generalized sense as well as the space $C[a,b]$ of all real valued continuous functions on the compact interval $[a,b]$ endowed with the "sup norm."

Proposition 3.1:

Let X in the generalized sense be a uniformly convex Banach space, then for any $\bigcup_{i=1}^{n}\delta_i>0,\bigcup_{i=1}^{n}\varepsilon_i>0$ and arbitrary vectors $\bigcup_{i=1}^{n}x_i,\bigcup_{j=1}^{n}x_j\in X$ with $\bigcup_{i=1}^{n}\|x_i\|\le\bigcup_{i=1}^{n}\delta_i,\bigcup_{j=1}^{n}\|x_j\|\le\delta_i,\bigcup_{\substack{i=1\\j=1}}^{n}\|x_i-x_j\|\ge\bigcup_{i=1}^{n}\varepsilon_i$

there exists a δ >0 such that

$$\bigcup_{\substack{i=1\\j=1}}^{n}\|(x_i+x_j)\|\le\bigcup_{i=1}\left[1-\delta_i\left(\frac{\varepsilon_i}{\delta_i}\right)\right]\delta_i$$

Proof:

For arbitrary $\bigcup_{i=1}^{n}x_i,\bigcup_{j=1}^{n}x_j\in X$, let $\bigcup_{i=1}^{n}z_i=\frac{x_i}{d_i}$, $\bigcup_{j=1}^{n}z_j=\frac{x_j}{d_j}$ and set $\bigcup_{i=1}^{n}\varepsilon_i=\bigcup_{i=1}^{n}\left(\frac{\varepsilon_i}{\delta_i}\right)$ obviously, $\bigcup\varepsilon_i>0$.

Moreover $\bigcup_{i=1}^{n}\|z_i\|\le1,\bigcup_{j=1}^{n}\|z_j\|\le1$ and

$$\bigcup_{\substack{i=1\\j=1}}^{n}\|z_i-z_j\|=\bigcup_{\substack{i=1\\j=1}}^{n}\left(\frac{i}{d_i}\|x_i-x_j\|\right)\ge\bigcup_{i=1}^{n}\left(\frac{\varepsilon_i}{\delta_i}\right)=\varepsilon$$

so that for generalized uniform convexity, we have, for some $\bigcup_{i=1}^{n}\delta_i=\bigcup_{i=1}^{n}\delta_i\left(\frac{\varepsilon_i}{\delta_i}\right)>0$

$$\bigcup_{\substack{i=1\\j=1}}^{n}\left\|\frac{1}{2}(z_i-z_j)\right\|\le\bigcup_{i=1}^{n}(1-\delta_i(\varepsilon))$$

that is

$$\bigcup_{\substack{i=1\\j=1}}^{n}\left\|\frac{1}{2d_i}(x_i+x_j)\right\|\le\bigcup_{i=1}^{n}\left(1-\delta_i\left(\frac{\varepsilon}{\delta}\right)\right)\delta$$

which implies

$$\bigcup_{\substack{i=1\\j=1}}\left\|\frac{1}{2}(x_i+x_j)\right\|\le\left(1-\delta\left(\frac{\varepsilon}{\delta}\right)\delta\right)$$

Proposition 3.2:

Let X in the general sense be a uniformly convex space and let $\alpha\in(0,1)$ and $\bigcup_{i=1}^{n}\varepsilon>0$. Then, for any $\bigcup_{i=1}^{n}\delta>0$, if x_i, x_j ∈ X are such that $\bigcup_{i=1}^{n}\|x_i\|\le\delta_i,\bigcup_{j=1}^{n}\|x_j\|\le\delta_j,\bigcup_{\substack{i=1\\j=1}}^{n}\|x_i-x_j\|\ge\bigcup_{i=1}^{n}\varepsilon_i$, then there exists $\bigcup_{i=1}^{n}\delta_i=\bigcup_{i=1}^{n}\delta_i\left(\frac{\varepsilon_i}{\delta_i}\right)>0$ such that

$$\bigcup_{\substack{i=1\\j=1}}\|\alpha x_i+(1-\alpha)x_j\|\le\bigcup_{i=1}\left\{\begin{bmatrix}1-2\delta\left(\frac{\varepsilon_i}{\delta_i}\right)\min\alpha,\\(1-\alpha)\end{bmatrix}\delta\right\}$$

Theorem 3.2:

Every generalized uniformly convex space is strictly convex.

Definition 3.6 (Convex Function in the Generalized sense)

A function $\bigcup_{i=1}^{n}f_i:\mathbb{R}^n\to\mathbb{R}$ is said to be convex in the generalized sense if its domain $D\left(\bigcup_{i=1}^{n}f_i\right)$ is a convex set and for every $\bigcup_{i=1}^{n}x_i,\bigcup_{j=1}^{n}x_j\in D\left(\bigcup_{i=1}^{n}f_i\right)$,

$$\bigcup_{\substack{i=1\\j=1}}^{n}f_i\left[\lambda x_i+(1-\lambda)x_j\right]\le\bigcup_{i=1}\lambda f_i(x_i)+\bigcup_{\substack{i=1\\j=1}}(1-\lambda)f_i(x_j)$$

Where $0\le\lambda\le1$.

Lemma 3.3: Every generalized convex function $\bigcup_{i=1}^{n}f_i$ with convex domain in $\mathbb{R}$ is continuous.

Definition 3.6:

Let X is a normed space with dim $X \geq 2$. The generalized modulus of convexity of X is the generalized function

$$\bigcup_{i=1}^{n} \delta_x : (0,2] \rightarrow [0,1]$$

defined by

$$\bigcup_{i=1}^{n} \delta_x(\varepsilon_i) = \inf \left\{ \begin{array}{l} 1 - \bigcup_{\substack{i=1 \\ j=1}}^{n} \left\| \frac{x_i + x_j}{2} \right\| : \bigcup_{i=1}^{n} \|x_i\| = \bigcup_{j=1}^{n} \|x_j\| = 1; \\ \bigcup_{i=1}^{n} \varepsilon_i = \bigcup_{\substack{i=1 \\ j=1}} \|x_i - x_j\| \end{array} \right\}$$

where in particular for an inner product H, we have

$$\bigcup_{i=1}^{n} \delta_H(\varepsilon_i) = 1 - \sqrt{1 - \frac{\bigcup_{i=1}^{n} \varepsilon_i^2}{4}}$$

Lemma 3.4: Let X is a normed space with dim $X \geq 2$. Then,

$$\delta_x(\varepsilon_i) = \inf \left\{ \begin{array}{l} 1 - \bigcup_{\substack{i=1 \\ j=1}}^{n} \left\| \frac{x_i + x_j}{2} \right\| ; \bigcup_{i=1}^{n} \|x_i\| \leq 1; \bigcup_{j=1}^{n} \|x_j\| \leq 1; \\ \bigcup_{i=1}^{n} \varepsilon_i \leq \bigcup_{\substack{i=1 \\ j=1}}^{n} \|x_i - x_j\| \end{array} \right\}$$

$$= \inf \left\{ \begin{array}{l} 1 - \bigcup_{\substack{i=1 \\ j=1}}^{n} \left\| \frac{x_i + x_j}{2} \right\| ; \bigcup_{i=1}^{n} \|x_i\| \leq 1; \bigcup_{j=1}^{n} \|x_j\| \leq 1; \\ \bigcup_{i=1}^{n} \varepsilon_i \leq \bigcup_{\substack{i=1 \\ j=1}} \|x_i - x_j\| \end{array} \right\}$$

This lemma implies $\bigcup_{i=1}^{n} \delta_x(0) = 0$.

Lemma 3.5: For every normed space X, the generalized function $\bigcup_{i=1}^{n} \left[\delta_x(\varepsilon_i) / \varepsilon_i \right]$ is decreasing on (0,2].

Theorem 3.3: The general modulus of convexity of a normed space X, δ_x is a generalized convex and continuous function.

Theorem 3.4: A normed space X is a generalized uniformly convex spaces of $\delta_x : \bigcup_{i=1}^{n} (\varepsilon_i) > 0$ for all $\varepsilon_i \in (0,2]$

Theorem 3.5: If X is an arbitrary uniformly convex space in the generalized sense, then

$$\bigcup_{i=1}^{n} \delta_x(\varepsilon_i) \leq 1 - \sqrt{\frac{\bigcup_{i=1}^{n} \varepsilon_i^2}{4}}$$

Theorem 3.6: (Generalized Milman Pettis theorem)

If X is a uniformly convex Banach space in the generalized sense, then X is reflexive in the generalized sense.

Proof of Theorem 3.1

Recall the parallelogram law in its generalized sense because this will be useful in our proof. Hence, for each $\bigcup_{i=1}^{n} x_i, \bigcup_{j=1}^{n} x_j \in H$, we have

$$\bigcup_{\substack{i=1 \\ j=1}}^{n} \|x_i + x_j\|^2 + \bigcup_{\substack{i=1 \\ j=1}}^{n} \|x_i - x_j\|^2 =$$

$$2\left(\bigcup_{i=1}^{n} \|x_i\|^2 + \bigcup_{j=1}^{n} \|x_j\|^2 \right) \tag{3.1.1}$$

Let $\bigcup_{i=1}^{n} \varepsilon_i \in (0,2]$ be given and let $\bigcup_{i=1}^{n} x_i, \bigcup_{j=1}^{n} x_j \in H$ be such that $\bigcup_{i=1}^{n} \|x_i\| \leq 1, \bigcup_{j=1}^{n} \|x_j\| \leq 1$ and $\bigcup_{\substack{i=1 \\ j=1}}^{n} \|x_i - x_j\| \geq \bigcup_{i=1}^{n} \varepsilon_i$ then equation (3.1.1) yields

$$\bigcup_{\substack{i=1 \\ j=1}}^{n} \left\| \frac{1}{2}(x_i - x_j) \right\| \leq \frac{1}{4} \left[2(2) - \bigcup_{\substack{i=1 \\ j=1}}^{n} \|x_i - x_j\|^2 \right] =$$

$$1 - \bigcup_{\substack{i=1 \\ j=1}} \left\| \frac{1}{2}(x_i - x_j) \right\|^2 \leq 1 - \frac{1}{4} \varepsilon^2 \bigcup_{\substack{i=1 \\ j=1}}^{n} \left\| \frac{1}{2}(x_i - x_j) \right\|$$

$$\leq \sqrt{1 - \frac{1}{4} \varepsilon^2}$$

We can now choose $\bigcup_{i=1}^{n} \delta_i = 1 - \sqrt{1 - \frac{1}{4} \bigcup_{i=1}^{n} \varepsilon^2} > 0$

Proof of example 3.3:

Given $\bigcup_{i=1}^{n} \varepsilon_i \in (0,2]$, let $\bigcup_{i=1}^{n} x_i, \bigcup_{j=1}^{n} x_j \in L_p$ be such that $\bigcup_{i=1}^{n} \|x_i\| \leq 1, \bigcup_{j=1}^{n} \|x_j\| \leq 1$ and $\bigcup_{\substack{i=1 \\ j=1}}^{n} \|x_i - x_j\| \geq \bigcup_{i=1}^{n} \varepsilon_i$, two cases arises:

Case 1: $1<p\leq 2$: In this case, the Helly's theorem yields

$$\bigcup_{\substack{i=1\\j=1}}^{n}\left\|\frac{1}{2}\left(x_i+x_j\right)\right\|^q+\bigcup_{\substack{i=1\\j=1}}^{n}\left\|\frac{1}{2}\left(x_i-x_j\right)\right\|^q\leq 2^{-(q-1)}$$

$$\left(\bigcup_{i=1}^{n}\|x_i\|^p+\bigcup_{j=1}^{n}\|x_j\|^p\right)^{q-1}\leq 2^{-(q-1)}2^{(q-1)}=1$$

Thus,

$$\bigcup_{\substack{i=1\\j=1}}^{n}\left\|\frac{1}{2}\left(x_i+x_j\right)\right\|^q$$

$$\leq 1-\bigcup_{\substack{i=1\\j=1}}^{n}\left\|\frac{x_i-x_j}{2}\right\|\leq 1-\bigcup_{i=1}\left(\frac{\varepsilon_i}{2}\right)^q\bigcup_{\substack{i=1\\j=1}}^{n}\left\|\frac{1}{2}\left(x_i+x_j\right)\right\|$$

$$\leq\bigcup_{\substack{i=1\\j=1}}\left[1-\left(\frac{\varepsilon_i}{2}\right)^q\right]^{1/q}$$

So that by choosing

$$\bigcup_{i=1}\delta_i=1-\bigcup_{i=1}\left[1-\left(\frac{\varepsilon_i}{2}\right)^q\right]^{1/q}>0$$

We obtain $\bigcup_{\substack{i=1\\j=1}}^{n}\left\|\frac{1}{2}\left(x_i-x_j\right)\right\|\leq\bigcup_{i=1}\left(1-\delta_i\right)$ and so L_p ($1<p\leq 2$) is uniformly convex in the generalized sense.

Case 2: $2\leq p<\infty$. The result follows as in case 1.

Proof of example 3.2 verification:

To see this follow the steps below:

For the space ℓ_1: Consider

$$\bigcup_{i=1}^{n}x_i=\left(1,0,0,0,\cdots\right)$$

$$\bigcup_{j=1}^{n}x_j=\left(0,-1,0,0,\cdots\right)$$

and take $\bigcup_{i=1}^{n}\varepsilon_i=1$ clearly $\bigcup_{i=1}^{n}x_i,\bigcup_{j=1}^{n}x_j\in L_1$ and

$\bigcup_{i=1}^{n}\|x_i\|_{\ell_1}=1,\bigcup_{j=1}^{n}\|x_j\|_{\ell_1}=1$ also while

$\bigcup_{\substack{i=1\\j=1}}^{n}\|x_i-x_j\|_{\ell_1}=2>\varepsilon.$ However,

$\bigcup_{\substack{i=1\\j=1}}^{n}\left\|\frac{1}{2}\left(x_i+x_j\right)\right\|=1$ so that

$$\bigcup_{\substack{i=1\\j=1}}^{n}\left\|\frac{1}{2}\left(x_i+x_j\right)\right\|<\bigcup_{i=1}^{n}\left(1-\delta_i\right),\bigcup_{i=1}^{n}\delta_i>0$$

is not satisfied showing that ℓ_1 is not uniformly convex in the generalized sense.

For the space ℓ_∞: Consider

$$\bigcup_{i=1}^{n}x_i=\left(1,1,0,0,\cdots\right)$$

$$\bigcup_{j=1}^{n}x_j=\left(-1,1,0,0,\cdots\right)$$

and take $\bigcup_{i=1}^{n}\varepsilon_i=1$ clearly $\bigcup_{i=1}^{n}x_i,\bigcup_{j=1}^{n}x_j\in\ell_\infty$ and

$\bigcup_{i=1}^{n}\|x_i\|_{\ell_\infty}=1,\bigcup_{j=1}^{n}\|x_j\|_{\ell_\infty}=1$ also while

$\bigcup_{\substack{i=1\\j=1}}^{n}\|x_i-x_j\|_{\ell_\infty}=2>\varepsilon.$ However,

$\bigcup_{\substack{i=1\\j=1}}^{n}\left\|\frac{1}{2}\left(x_i+x_j\right)\right\|=1$ so that ℓ_∞ is not uniformly convex in the generalized sense.

For the space $C[a,b]$: we choose two function $\bigcup_{i=1}^{n}f_i$ and $\bigcup_{j=1}^{n}f_j$ as follows

$\bigcup_{i=1}^{n}f_i\left(t_i\right)=1,$ for all $t\in[a,b]$, $f_j\left(t_j\right)=\frac{b-t_j}{b-a}$, for each $t\in[a,b]$.

Take $\bigcup_{i=1}^{n}\varepsilon_i=\frac{1}{2}$ clearly $\bigcup_{i=1}^{n}f_i$ and $\bigcup_{j=1}^{n}f_j\in C\left[a,b\right]$,

$\bigcup_{i=1}^{n}\|f_i\|=\bigcup_{j=1}^{n}\|f_j\|=1$ and $\bigcup_{\substack{i=1\\j=1}}^{n}\|f_i-f_j\|=1>\varepsilon$. Also

$\bigcup_{\substack{i=1\\j=1}}^{n}\left\|\frac{1}{2}\left(f_i-f_j\right)\right\|=1$ and so $C[a,b]$ is not uniformly convex.

Proof of proposition 3.1:

Let $\bigcup_{i=1}\varepsilon_i>0$ be given and let $\bigcup_{i=1}^{n}z_i=\bigcup_{i=1}^{n}\left(\frac{x_i}{d_i}\right),\bigcup_{j=1}^{n}z_j=\bigcup_{j=1}^{n}\left(\frac{x_j}{d_j}\right)$ and suppose we set $\bigcup_{i=1}^{n}\varepsilon_i=\bigcup_{i=1}^{n}\left(\frac{\varepsilon_i}{d_i}\right)$. Obviously, moreover $\bigcup_{i=1}^{n}\|z_i\|\le 1,\bigcup_{j=1}^{n}\|z_j\|\le 1$ and $\bigcup_{\substack{i=1\\j=1}}^{n}\|z_i-z_j\|=\bigcup_{\substack{i=1\\j=1}}\frac{1}{d_i}\|x_i-x_j\|\ge\frac{\varepsilon}{d}=\varepsilon$. Now by the generalized uniform convexity, we have

$$\bigcup_{\substack{i=1\\j=1}}^{n}\left\|\frac{1}{2}\left(z_i+z_j\right)\right\|\le\bigcup_{i=1}^{n}\left(1-\delta_i\left(\varepsilon_i\right)\right)$$

$$\bigcup_{\substack{i=1\\j=1}}^{n}\left\|\frac{1}{2d}\left(x_i+x_j\right)\right\|\le\bigcup_{i=1}^{n}\left(1-\delta_i\left(\frac{\varepsilon_i}{d_i}\right)\right)$$

which implies

$$\bigcup_{\substack{i=1\\j=1}}^{n}\left\|\frac{1}{2}\left(x_i+x_j\right)\right\|\le\bigcup_{i=1}^{n}\left[1-\delta_i\left(\frac{\varepsilon_i}{d_i}\right)\right]$$

Proof of Theorem 3.6 (Milman Pettis)

It suffices to prove that map $\bigcup_{i=1}^{n}J_i:B_X\to B_{X^{**}}$ is subjective. So let $\bigcup_{i=1}^{n}\varepsilon_i\in\bigcup_{i=1}^{n}J_i\left(B_X\right)$. Recall that $\bigcup_{i=1}^{n}\overline{J_i\left(B_X\right)}=\bigcup_{i=1}^{n}J_i\left(B_X\right)$ hence it suffices to prove $\bigcup_{i=1}^{n}\varepsilon_i\in\bigcup_{i=1}^{n}\overline{J_i\left(B_X\right)}$. To prove this, it suffices to show that any open ball with center $\bigcup_{i=1}^{n}\varepsilon_i$ intersects $\bigcup_{i=1}^{n}J_i\left(B_X\right)$ i.e. given any ε>0 $\bigcup_{i=1}^{n}B_\varepsilon\left(\varepsilon_i\right)\bigcup_{i=1}^{n}J_i\left(B_X\right)\ne 0$ or better still $\bigcup_{i=1}^{n}\varepsilon_i>0,\exists\bigcup_{i=1}^{n}x_i\in B_X$ such that

$$\bigcup_{i=1}^{n}\left\|\varepsilon_i-J_i\left(x_i\right)\right\|<\varepsilon \qquad (3.6.1)$$

We now prove (3.6.1). So let $\bigcup_{i=1}^{n}\varepsilon_i$ >0 be given by the uniform convexity of X in the general sense there exists $\bigcup_{i=1}^{n}\delta_i>0$ such that for all $\bigcup_{i=1}^{n}x_i,\bigcup_{j=1}^{n}x_j\in X$ with $\bigcup_{i=1}^{n}\|x_i\|\le 1,\bigcup_{j=1}^{n}\|x_j\|\le 1,\bigcup_{\substack{i=1\\j=1}}^{n}\|x_i-x_j\|\ge\bigcup_{i=1}^{n}\varepsilon_i$ we have

$$\bigcup_{\substack{i=1\\j=1}}^{n}\left\|\frac{1}{2}\left(x_i-x_j\right)\right\|<\bigcup_{i=1}^{n}\left(1-\delta_i\right)$$

Fix this $\bigcup_{i=1}^{n}\delta_i>0$ (corresponding to the given $\bigcup_{i=1}^{n}\varepsilon_i$)

Since $\bigcup_{i=1}^{n}\|\varepsilon_i\|=1$, it follows that $\bigcup_{i=1}^{n}\varepsilon_i\ne 0$. Hence, we can choose $\bigcup_{i=1}^{n}f_i\in B$ such that $\bigcup_{i=1}^{n}\|f_i\|=1$ and

$$\bigcup_{i=1}^{n}f_i\left(\varepsilon\right)=\left\langle\bigcup_{i=1}^{n}\varepsilon_i,\bigcup_{i=1}^{n}f_i\right\rangle \qquad (3.6.2)$$

$$=\bigcup_{i=1}^{n}\|\varepsilon_i\|=1>\bigcup_{i=1}^{n}\left(1-\frac{\delta_i}{2}\right)$$

Set

$$X=\left\{\bigcup_{i=1}^{n}x_i\in X^{**}:\left|\left\langle\bigcup_{i=1}^{n}\left(x_i-\varepsilon_i\right),\bigcup f_i\right\rangle\right|<\frac{\bigcup\delta_i}{2}\right\}.$$

Then, X is a neighborhood of $\bigcup_{i=1}^{n}\varepsilon_i$, the w^* topology of X^{**} so $\bigcup_{i=1}^{n}J_i\left(B_X\right)$ is sense in $B_{X^{**}}$ with respect to the topology w^* of X^{**}. Hence, any neighborhood of the w^* topology of X^{**} an arbitrary element of $B_{X^{**}}$ must not intersect $\bigcup_{i=1}^{n}J_i\left(B_X\right)$. In particular

$$X\cap\left(\bigcup_{i=1}^{n}J_i\left(B_X\right)\right)\ne 0$$

Let $\bigcup_{i=1}^{n}x_j\in X\cap\left(\bigcup_{i=1}^{n}J_i\left(B_X\right)\right)$ then clearly $\bigcup_{i=1}^{n}x_j\in\bigcup_{i=1}^{n}J_i\left(B_X\right)$. Let $x_0\in B_x$ be such that $\bigcup_{i=1}^{n}J_i\left(B_X\right)=X$ then $\bigcup_{i=1}^{n}J_i\left(B_X\right)\in X$. We have thus picked $x_0\in B_x$ such that $\bigcup_{i=1}^{n}J_i\left(B_X\right)\in X$.

.Then, to complete the proof of (3.6.1), it now suffices to prove that

$$\bigcup_{i=1}\left\|\varepsilon_i - J_i\left(x_0\right)\right\| \le \varepsilon$$

We now prove (3.6.3) by contradiction. So suppose then

$$\bigcup\varepsilon_i \notin \bigcup_{i=1}^{n} J_i\left(x_0\right)+\varepsilon_i B_{X^{**}}$$

Then,

$$\bigcup_{i=1}^{n}\varepsilon_i \in \bigcup_{i=1}^{n} J_i\left(x_0\right)+\varepsilon_i B_{X^{**}} := Y$$

the complement of $\left(\bigcup_{i=1}^{n} J_i\left(x_0\right)+\varepsilon_i B_{X^{**}}\right)$.

observe that since $B_{x^{**}}$ is closed, it follows that Y is a neighborhood of $\bigcup_{i=1}^{n}\varepsilon_i$ in the w^* w^* topology of X^*. Hence,

$$\left(X\cap Y\right)\cap\left(\bigcup_{i=1}^{n} J_i\left(B_X\right)\right)$$

This implies that there exists $\bigcup_{i=1}^{n} x_i \in B_X$ such that

$$\bigcup_{i=1}^{n} J_i\left(x_i\right)\in X\cap Y$$

Consequently, we obtain

$$\left|\left\langle \bigcup_{i=1}^{n} J_i\left(x_0\right)-\bigcup_{i=1}^{n}\varepsilon_i,\bigcup_{i=1}^{n} f_i\right\rangle\right| = \left|\begin{array}{l}\left\langle \bigcup_{i=1}^{n} J_i\left(x_0\right),\bigcup_{i=1}^{n} f_i\right\rangle - \\ \left\langle \bigcup_{i=1}^{n}\varepsilon_i,\bigcup_{i=1}^{n} f_i\right\rangle\end{array}\right|$$

$$=\left|\left\langle \bigcup_{i=1}^{n} f_i, x_0\right\rangle - \left\langle \bigcup_{i=1}^{n}\varepsilon_i,\bigcup_{i=1}^{n} f_i\right\rangle\right|$$

(since we already noted that $\bigcup_{i=1}^{n} J_i\left(x_0\right)\in X$) and

$$\left|\left\langle \bigcup_{i=1}^{n} f_i,\bigcup_{i=1}^{n} x_i\right\rangle - \left\langle \bigcup_{i=1}^{n}\varepsilon_i,\bigcup_{i=1}^{n} f_i\right\rangle\right|$$

$$<\bigcup_{i=1}^{n}\frac{\delta_i}{2}\left(\text{since } \bigcup_{i=1}^{n} J_i\left(x\right)\in X\right)$$

These inequalities imply

$$\left|2\left\langle \bigcup_{i=1}^{n}\varepsilon_i,\bigcup_{i=1}^{n} f_i\right\rangle - \left\langle \bigcup_{i=1}^{n} f_i,\bigcup_{i=1}^{n}\left(x_0+x_1\right)\right\rangle\right|$$

$$\le\left|\left\langle \bigcup_{i=1}^{n} f_i, x_0\right\rangle - \left\langle \bigcup_{i=1}^{n}\varepsilon_i,\bigcup_{i=1}^{n} f_i\right\rangle\right|$$

$$+\left|\left\langle \bigcup_{i=1}^{n} f_i,\bigcup_{i=1}^{n} x_i\right\rangle - \left\langle \bigcup_{i=1}^{n}\varepsilon_i,\bigcup_{i=1}^{n} f_i\right\rangle\right| \le \delta$$

So that

$$2\left\langle \bigcup_{i=1}^{n}\varepsilon_i,\bigcup_{i=1}^{n} f_i\right\rangle \le \left|\left\langle \bigcup_{i=1}^{n} f_i,\bigcup_{i=1}^{n}\left(x_0+x_i\right)\right\rangle\right|$$

$$+\delta \le \left\|\bigcup_{i=1}^{n} f_i\right\|.\left\|\bigcup_{i=1}^{n}\left(x_0+x_i\right)\right\|+\delta \le \left\|\bigcup_{i=1}^{n}\left(x_0+x_i\right)\right\|+\delta$$

and consequently using (3.6.2), we obtain the following estimate

$$\left\|\frac{1}{2}\bigcup_{i=1}^{n}\left(x_0+x_i\right)\right\| \ge \left\langle \bigcup_{i=1}^{n}\varepsilon_i,\bigcup_{i=1}^{n} f_i\right\rangle - \bigcup_{i=1}^{n}\frac{\delta}{2} > 1 -$$

$$\bigcup_{i=1}^{n}\frac{\delta}{2} - \bigcup_{i=1}^{n}\frac{\delta}{2} = 1 - \bigcup_{i=1}^{n}\delta_i \le \left\|\bigcup_{i=1}^{n}\left(x_0+x_i\right)\right\|+\delta$$

We have the following situation $\left\|x_0\right\| \le \bigcup_{i=1}^{n}\left\|x_i\right\| \le 1$ and $\bigcup_{i=1}^{n}\left\|\frac{1}{2}\left(x_0+x_i\right)\right\| > \bigcup\left(1-\delta_i\right)$ then by uniform convexity (contrapositive), we have

$$\bigcup_{i=1}^{n}\left\|\frac{1}{2}\left(x_0+x_i\right)\right\| \le \bigcup_{i=1}^{n}\varepsilon_i = \varepsilon$$

But, then $\bigcup_{i=1}^{n}\left\|\frac{1}{2}\left(x_0+x_i\right)\right\| > \varepsilon$ since $\bigcup_{i=1}^{n} J_i\left(x\right)\in Y$ and $\bigcup_{i=1}^{n} J_i\left(x_0\right)\notin Y$ which implies

$$\bigcup_{i=1}^{n}\left\|x_0-x_i\right\| = \bigcup\left\|J_i\left(x_0-x_i\right)\right\| = \bigcup\left\|J_i x_0 - J_i x_i\right\| > \varepsilon$$

hence contradiction.

Proof of Lemma 3.5

Fix $0<\bigcup_{i=1}^{n}\eta_i \le 2$ with $\bigcup_{i=1}^{n}\eta_i \le \bigcup_{i=1}^{n}\varepsilon_i$ and $\bigcup_{i=1}^{n} x_i,\bigcup_{j=1}^{n} x_j$ in X such that $\bigcup_{i=1}^{n}\left\|x_i\right\| = 1 = \bigcup_{j=1}^{n} x_j$ and $\bigcup_{\substack{i=1\\j=1}}^{n}\left\|x_i - x_j\right\| = \bigcup_{i=1}^{n}\varepsilon_i$ suffices to prove

$\bigcup_{i=1}^{n}\left(\frac{\delta_X(\eta)}{\eta}\right) \leq \bigcup_{i=1}^{n}\left(\frac{\delta_X(\varepsilon_i)}{\varepsilon_i}\right)$. Consider

$$\bigcup_{i=1}^{n} a_i = \bigcup_{i=1}^{n}\left(\frac{\eta_i}{\varepsilon_i} x_i\right) + \bigcup_{\substack{i=1\\j=1}}^{n}\left(1-\frac{\eta_i}{\varepsilon_i}\right)\left(\frac{x_i+x_j}{\|x_i+x_j\|}\right)$$

and

$$\bigcup_{j=1}^{n} a_j = \bigcup_{j=1}^{n}\left(\frac{\eta_j}{\varepsilon_j} x_j\right) + \bigcup_{\substack{i=1\\j=1}}^{n}\left(1-\frac{\eta_{ij}}{\varepsilon_j}\right)\left(\frac{x_i+x_j}{\|x_i+x_j\|}\right)$$

then

$$\bigcup_{i=1}^{n} a_i - \bigcup_{j=1}^{n} a_j = \bigcup_{\substack{i=1\\j=1}}^{n}\frac{\eta_i}{\varepsilon_i}(x_i+x_j), \left\|\bigcup_{i=1}^{n} a_j - \bigcup_{i=1}^{n} a_j\right\| = \eta_j$$

and

$$\left(\frac{\bigcup_{i=1}^{n} a_j - \bigcup_{i=1}^{n} a_j}{2}\right) = \bigcup_{\substack{i=1\\j=1}}^{n}\left(\frac{x_i+x_j}{\|x_i+x_j\|}\right)\left(\frac{\eta_i}{1-\frac{\eta_i}{\varepsilon_i}+\frac{\|x_i+x_j\|}{2\varepsilon_i}}\right)$$

which implies that

$$\bigcup_{\substack{i=1\\j=1}}^{n}\left\|\frac{x_i+x_j}{\|x_i+x_j\|} - \frac{a_i+a_j}{2}\right\| = \bigcup_{i=1}^{n}\left(\frac{\eta_i}{\varepsilon_i}\right) - \bigcup_{\substack{i=1\\j=1}}^{n}\left(\frac{\eta_i\|x_i+x_j\|}{2\varepsilon_i}\right)$$

$$= 1 - \bigcup_{\substack{i=1\\j=1}}^{n}\left(1-\frac{\eta_i}{\varepsilon_i}+\frac{\eta_i\|x_i+x_j\|}{2\varepsilon_i}\right)$$

$$= \bigcup_{\substack{i=1\\j=1}}^{n}\left(1-\frac{\|a_i+a_j\|}{2}\right)$$

Note that

$$\bigcup_{\substack{i=1\\j=1}}^{n}\left\|\frac{x_i+x_j}{\|x_i+x_j\|} - \frac{x_i+x_j}{2}\right\|$$

$$= \bigcup_{\substack{i=1\\j=1}}^{n}\|x_i+x_j\|\left(\frac{1}{\|x_i+x_j\|} - \frac{1}{2}\right)$$

$$= \bigcup_{\substack{i=1\\j=1}}^{n}\left(1-\frac{\|x_i+x_j\|}{2}\right) \qquad (3.5.1)$$

Now,

$$\bigcup_{\substack{i=1\\j=1}}^{n}\frac{\left\|\frac{x_i+x_j}{\|x_i+x_j\|} - \frac{a_i+a_j}{2}\right\|}{\|a_i-a_j\|}$$

$$= \bigcup_{\substack{i=1\\j=1}}^{n}\frac{1}{\eta_i}\left(\frac{\eta_i}{\varepsilon_i} - \frac{\eta_i\|x_i+x_j\|}{2\varepsilon_i}\right) = \bigcup_{\substack{i=1\\j=1}}\frac{1}{\varepsilon_i}\left(1-\frac{\|x_i+x_j\|}{2}\right)$$

$$= \bigcup\frac{\left\|\frac{x_i+x_j}{\|x_i+x_j\|} - \frac{x_i+x_j}{2}\right\|}{\|x_i-x_j\|} \qquad (3.5.2)$$

and then

$$\bigcup_{i=1}^{n}\left(\frac{\delta_X(\eta_i)}{\eta_i}\right) \leq$$

$$\bigcup_{\substack{i=1\\j=1}}^{n}\left(\frac{1-\frac{\|a_i+a_j\|}{2}}{\|a_i+a_j\|}\right) = \bigcup_{\substack{i=1\\j=1}}^{n}\frac{\frac{x_i+x_j}{\|x_i+x_j\|} - \frac{\|a_i+a_j\|}{2}}{\|a_i+a_j\|}$$

$$= \bigcup_{\substack{i=1\\j=1}}^{n}\frac{\left\|\frac{x_i+x_j}{\|x_i+x_j\|} - \frac{x_i+x_j}{2}\right\|}{\|x_i+x_j\|}$$

$$= \bigcup_{\substack{i=1\\j=1}}^{n}\frac{\left(1-\frac{\|x_i+x_j\|}{2}\right)}{\|x_i-x_j\|} = \bigcup_{\substack{i=1\\j=1}}^{n}\left(\frac{1-\frac{\|x_i+x_j\|}{2}}{\varepsilon_i}\right)$$

By taking the infimum over all possible $\bigcup_{i=1}^{n} x_i$ and $\bigcup_{j=1}^{n} x_j$ with $\bigcup_{i=1}^{n}\varepsilon_i = \bigcup_{\substack{i=1\\j=1}}^{n}\|x_i - x_j\|$ and

$$\bigcup_{i=1}^{n}\|x_i\| = \bigcup_{j=1}^{n}\|x_j\| = 1$$

we obtain that $\bigcup_{i=1}^{n}\frac{\delta_X(\eta_i)}{\eta_i} \leq \bigcup_{i=1}^{n}\frac{\delta_X(\varepsilon_i)}{\varepsilon_i}$.

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