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On the Analytic Review of Some Contraction and Extension Results in the Hilbert Space with Applications in the Solution of Some Elasticity Problems

Eziokwu C. Emmanuel

Department of Mathematics, Micheal Okpara University of Agriculture, Umudike, Abia State, Nigeria

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ABSTRACT

This research reviews the analysis of contraction and extension maps in the Hilbert space through the spectral theory approach. Some very important results were discussed and illustrated fully the aid of which analytical works on contraction and extension was fully utilized in the applied mathematics of elasticity theory of various deformation problems as seen in section three of this work.

Key words: Hilbert space, Operators, Contraction, Extension, Spectra, Deformation

INTRODUCTION

In this work, we[1,2] make use of the normal, particularly self adjoint and unitary operators in Hilbert space. However, since less is known of the structure of normal operators for lack of satisfactory generalization, we, therefore, resort to finding relations that will reduce the problem of dealing with general linear operators to a more workable particular case of normal operators of which its simplest types are

$T=A+iB,$

where the bounded linear operator T in the Hilbert space. $\mathfrak{H}$ is represented by the two self-adjoint operators

$$A = ReT = \frac{1}{2}\left(T + T^*\right),\ B = \operatorname{Im}T - \frac{1}{2i}\left(T - T^*\right),$$

And

$T=VR$

where the bounded linear isometric operator (which, in certain cases can be chosen to be unitary, in particular if T is a one-to-one operator of the space $\mathfrak{H}$ onto itself). The applicability of these relations is restricted by the fact that neither A and B nor V and R are in general permutable, and there is no simple relation among the corresponding representations of the iterated operators T, T^2,…

In the sequel, we shall deal with other relations which are connected with extensions of a given operator. However, contrary to what we usually do, we shall also allow extensions which extend beyond the given space.

Hence, by an extension of a linear operator T of Hilbert space $\mathfrak{H}$, we shall understand a linear operator Tin a Hilbert space $\mathfrak{H}$ which contains H as a (not necessarily proper) subspace, such that $D_T \supseteq D_T$ and $\boldsymbol{T}f=Tf$ for $f \in D_T$. We shall retain notation $\boldsymbol{T} \supset T$ which we used for ordinary extensions (where $H=\mathfrak{H}$).

The orthogonal projection of the extension space H onto its subspace $\mathfrak{H}$ will be denoted by P_H or simply by P. Among the extensions of a bounded linear operator T in $\mathfrak{H}$ (with $D_T=H$), we shall consider in particular those which are of the form PS where S is bounded linear transformation of an extension space H. We express this relation

$T \subseteq PS$

by saying that T is the projection of the operator onto $\mathfrak{H}$, in symbols

Address for correspondence:
Eziokwu C. Emmanuel
E-mail: okereemm@yahoo.com

$$T = pr_{\mathfrak{H}}S \quad \text{or simply} \quad T = prS \qquad (1.1)$$

It is obvious that the relations $T_i = prS_i$ (i=1,2) imply the relation

$$a_1 T_1 + a_2 T_2 = pr(a_1 S_1 + a_2 S_2) \qquad (1.2)$$

(of course, S_1 and S_2 are operator in the same extension space H). Relation (1) also implies that

$$T^* = prS^* \qquad (1.3)$$

Finally, the uniform, strong, or weak convergence of a sequence $\{S_n\}$ implies convergence of the same type for the sequence $\{T_n\}$ where $T_n = prS_n$. If H and H' are two extension spaces of the same space H, S and S' are bounded linear operator of H and H', respectively, then we shall say that the structures {H, S, and H} and {H', S', and H} are isomorphic if H can be mapped isometrically onto H' in such a way that the elements of the common subspace H are left invariant and that $f \to f'$ implies $Sf \to S' f'$. If $\{S\omega\}_{\omega}\in\Omega$ and $\{S_{\omega}'\}(_{\omega}\in_{\Omega})$ are two families of bounded linear transformations in H and H^' respectively, we define the isomorphism of the structures.
The terminology T is the compression of S in $\mathfrak{H}$, and S is the dilation of T to H according to the proof by HALMOS.
In fact, we have

$$\left(T_0, T^* g\right)\left(TT_0, g\right)\left(PSPT_0, g\right) =$$
$$\left(T_0, PS^* Pg\right) = \left(T_0, PS^* g\right) \text{ for } T_0, g \in \mathfrak{H}$$

and

$$T_n - T_m \leqq S_n - S_m, \left(T_n - T_m\right)T_0 \leqq \left(S_n - S_m\right)T_0$$
$$\text{for } T_0 \in \mathfrak{H}$$

and

$$\left(\left(T_n - T_m\right)T_0, g\right) = \left(\left(S_n - S_m\right)T_0, g\right) \text{ for } T_0, g \in \mathfrak{H}.$$

$\left(H.S_{\omega}, \mathfrak{H}\right)_{\omega\in\Omega}$ and $\left\{H', S_{\omega}', \mathfrak{H}\right\}_{\omega\in\Omega}$ in the same manner by requiring that $T_0 \to T^*$ imply $S_{\omega}T_0 \to S_{\omega}' T^*$ for all $\omega \in \Omega$.

It is obvious that, from the point of view of extensions of operators in $\mathfrak{H}$ which extend beyond $\mathfrak{H}$, two extensions which give rise to two isomorphic structures can be considered as identical. In the sequel, when speaking of Hilbert spaces, we shall mean both real and complex spaces. If we wish to distinguish between real and complex spaces, we shall say so explicitly. Of course, an extension space H of is always of the same type (real or complex) as $\mathfrak{H}$.

Theorem 1.1:[1,2] A necessary and sufficient condition for the operator g given in a set E of the space C to be extendable to the entire space C so as to define there a linear operator of norm ≤M is that

$$\left\| \sum_{k=1}^{n} c_k G T_0 \right\| \leq M \left\| \sum_{k=1}^{n} c_k T_0 \right\| \qquad (1.4)$$

for every linear combination of remnants of E
Proof: See Krantz[3]

Generalized Spectral Families and the NEUMARK'S Theorem[2,4]

In extensions which extend beyond the given space M.A Neumark; he centers on self-adjoint extensions of symmetric operators in particular and consequently, if S is a symmetric operator in the complex Hilbert space $\mathfrak{H}$ (with D_S dense in $\mathfrak{H}$), we know that S cannot be extended to a self-adjoint operator without extending beyond $\mathfrak{H}$ except when the deficiency indices m and n of S are equal. On the other hand, there always exist self-adjoint extensions of S if one allows these extensions to extend beyond the space $\mathfrak{H}$.

This is easily proved: Choose, in a Hilbert space , a symmetric operator S' but in reverse order. One can take for example $\mathfrak{H}' = \mathfrak{H}$ and $S' = -S$. Having done this, we consider the product space $H = \mathfrak{H} \times \mathfrak{H}'$ whose elements are pairs $\left\{T, T'\right\}\left(T \in \mathfrak{H}, T' \in \mathfrak{H}\right)$ and in which the vector operations and metric are defined as follows:

$$c\{T,T'\} = \{cT, cT'\}; \{T_1, T_1'\} + \{T_2, T_2'\}$$
$$= \{T_1 + T_2, T_1' + T_2'\};$$

$$(\{T_1, T_1'\}, \{T_2, T_2'\}) = (T_1, T_2) + (T_1', T_2').$$

If we identify the element T in $\mathfrak{H}$ with the element $\{T,0\}$ in H, we embed $\mathfrak{H}$ in H as a subspace of the latter. The operator

$S\{T,T'\}=\{ST,S'\,T'\}(T\in D_s,T'\in D_s)$

is then, as can easily be seen, a symmetric operator in H having deficiency indices $m+n$, $n+m$. Consequently, S can be extended, without extending beyond H, to a self-adjoint operator A in H. Since we have

$S\subseteqq S\subseteqq A$

(where the first extension is obtained by extension from $\mathfrak{H}$ to H), we obtain a self-adjoint extension A of S.

Let

$$A\int_{-\infty}^{\infty}\lambda dE_\lambda$$

be the spectral decomposition of A. We have the relations

$$\left(ST,g\right)=\left(AT,Pg\right)=\int_{-\infty}^{\infty}\lambda d\left(E_\lambda T,Pg\right)$$
$$=\int_{-\infty}^{\infty}\lambda d\left(PE_\lambda T,g\right),$$

$$\|ST\|^2=\|AT\|^2=\int_{-\infty}^{\infty}\lambda^2 d\left(E_\lambda T,T\right)$$
$$=\int_{-\infty}^{\infty}\lambda^2 d\left(PE_\lambda T,T\right).$$

For $T\in D_s, g\in\mathfrak{H}$. Setting

$$B_\lambda=prE_\lambda \tag{1.2.1}$$

We obtain a family $\{B_\lambda\}_{-\infty<\lambda<\infty}$ of bounded self-adjoint operators in the space $\mathfrak{H}$, which have the following properties:

a. $B_\lambda\leqq B_\mu$ for $\lambda<\mu$;

b. $B_{\lambda+0}=B_\lambda$;

c. $B_\lambda\to 0$ as $\lambda\to-\infty$; $B_\lambda\to I$ as $\lambda\to+\infty$.

Every one-parameter family of bounded self-adjoint operator has that these properties will be called a generalized spectral family. If this family consists of projections (which are then, as a consequence of a), mutually permutable, then we have an ordinary spectral family.

According to what we just proved, we can assign to each symmetric operator in $\mathfrak{H}$ a generalized spectral family $\{B_\lambda\}$ in such a way that the equations.

$$\left(ST,g\right)=\int_{-\infty}^{\infty}\lambda d\left(B_\lambda T,g\right),\ \|ST\|^2$$
$$=\int_{-\infty}^{\infty}\lambda^2 d\left(B_\lambda T,T\right) \tag{1.2.2}$$

are satisfied for $T\in D_s, g\in\mathfrak{H}$ (where the integral in the second equation can also converge for certain T which do not belong to D_s).

Theorem 1.2.1:[4,5] Every generalized spectral family $\{B_\lambda\}$ can be represented in the form (4), as the projection of an ordinary spectral family $\{E_\lambda\}$. One can even require the extension space H to be minimal in the sense that it be spanned by the elements of the form $E_\lambda T$ where $T\in\mathfrak{H}, -\infty<\lambda<\infty$; in this case, the structure $\{H, E_\lambda, H\}_{-\infty<\lambda<\infty}$ is determined to within an isomorphism. The detail of this proof is as in.[4,5]

We now obtain thus the following corollary.

Corollary 1.2.2:[4,5] Every self-adjoint transformation A in the Hilbert space $\mathfrak{H}$, such that $0\leqq A\leqq I$, can be represented in the form

$A=prQ$

where Q is a projection in an extension space H. In brief: A is the projection of a projection.

This corollary can also be proved directly without recourse to the general Neumark theorem. The following construction is due to E.A Michael.

Consider the product space $H=\mathfrak{H}\times\mathfrak{H}$; by identifying the element T in $\mathfrak{H}$ with the element $\{T, 0\}$ in H, we embed $\mathfrak{H}$ in H as a subspace of the latter. If we write the elements H as one-column matrices $\begin{pmatrix}T\\ g\end{pmatrix}$, then every bounded linear operator T in H can be represented in the form of a matrix

$$T=\begin{pmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{pmatrix} \tag{1.2.6}$$

Whose elements T_{ik} bounded linear operator in $\mathfrak{H}$. It is easily verified that the matrix addition and multiplication of the corresponding matrices correspond to the addition and multiplication of the operators. Moreover, relation (6) implies that

$$T^*=\begin{pmatrix}T_{11}^* & T_{12}^*\\ T_{21}^* & T_{22}^*\end{pmatrix}$$

Finally, we have

$T=prT$

If and only if

$T_n=T$

This done, we consider the operator $Q=\begin{pmatrix} A & B \\ B & I-A \end{pmatrix}$ with $B=\left[A(I-A)\right]^{\frac{1}{2}}$

It is clear that Q is self-adjoint and that $A=prQ$. It remains only to show that $Q^2=Q$, which is easily done by calculating the square of the matrix Q. The following theorem is another, less special, and consequence of the Neumark theorem.

Theorem 1.2.3:[5,6] Every finite or infinite sequence $\{A_n\}$ of bounded self-adjoint operators in the Hilbert space such that

$$A_n \geqq 0, \sum A_n = 1$$

Can be represented in the form

$$A_n = prQ_n \ (n=1,2,\ldots),$$

where $\{Q_n\}$ is sequence of projections of an extension space H for which

$$Q_n Q_m = 0 \ (m \neq n), \sum Q_n = I$$

In fact, one has only to apply the Neumark theorem to the generalized spectral family $\{B_\lambda\}$defined by

$$B_\lambda = \sum_{n \leqq \lambda} A_n$$

If $\{E_\lambda\}$ is an ordinary spectral family in a minimal extension space such that $B_\lambda = prE_\lambda$, the function E_λ of λ increases only at the points n where it has jumps

$$Q_n = E_n - E_{n-0};$$

These operators Q_n satisfy the requirements of the theorem. This theorem in its turn has the following theorem as a consequence.

Theorem 1.2.4:[6,7] Every finite or infinite sequence $\{T_n\}$ of bounded linear operators in the complex Hilbert space $\mathfrak{H}$ can be represented by means of a sequence $\{N_n\}$ of bounded normal operators in an extension space H in the form

$$T_n = prN_n \ (n=1,2,\ldots),$$

Where the N_n is pair-wise doubly permutable. If any of the operator T_n is self-adjoint, the corresponding N_n can also be chosen to be self-adjoint. We first consider the case where all the operator T_n is self-adjoint. If m_n and M_n are the greatest lower and least upper bounds of T_n, we set

$$A_n = \frac{1}{2^n\left(M_n - m_n + 1\right)}\left(T_n - m_n I\right)\left(n=1,2,\ldots\right);$$

Then, we obviously have

$$A_n \leqq 0, \sum_n A_n \leqq I$$

If we again set

$$A = I - \sum_n A_n$$

we obtain a sequence $A, A_1, A_2, \ldots$, of operators which satisfies the hypotheses of the preceding theorem and which consequently can be represented in the form

$$A_n = prQ_n \ (n=1,2,\ldots)$$

In terms of the projections Q_n, which are pairwise orthogonal (and consequently permutable). It follows that

$$T_n = prS_n \ (n=1,2,\ldots)$$

with

$$S_n = m_n I + 2^n \left(M_n - m_n + 1\right) Q_n,$$

where the operator S_n is self-adjoint and mutually permutable. The general case is reducible to the particular case of self-adjoint operators by replacing each operator T_n in the given sequence by the two self-adjoint operators Re T_n and Im T_n. In fact, since the representation

$$\mathrm{Re}\, \boldsymbol{T}_n = prS_{2n}, \mathrm{Im} T_n = prS_{2n+1}) \ (n=1,2,\ldots)$$

is possible by means of bounded self-adjoint pairwise permutable operators S_1, the representation

$$T_n = prN_n \ (n=1,2,\ldots)$$

follows from this by means of the normal pair-wise doubly permutable operators $N_n = S_{2n} + iS_{2n+1}$. For a self-adjoint T_n, we have $T_n = \mathrm{Re}\, T_n$, and we can then choose $S_{2n+1} = 0$ and hence $N_n = S_{2n}$.

Sequences of Moments

1. The following theorem is closely related to the theorem on extension

Theorem 1.3.1: [7,8] Suppose $\{A_n\}$ (n=0,1,…) is a sequence of bounded self-adjoint operators in the Hilbert space satisfying the following conditions

$$(a_M)\begin{cases} \quad for\ every\ polynomial \\ \quad a_0+a_1\lambda+a_2\lambda^2+\ldots+a_n\lambda^n \\ with\ real\ coefficients\ which\ assumes\ non \\ -negative\ values\ in\ the\ interval \\ \quad -M\leqq\lambda\leqq M, we\ have \\ \quad a_0A_0+a_1A_1+a_2A_2+\ldots+a_nA_n\geqq 0 \end{cases}$$

$A_0=I.$ (1.3.1)

Then, there exists a self-adjoint operator A in an extension space H such that

$A_n=prA^n$ (n=0,1,…,) (1.3.2)

The proof of this theorem is based on the concept of the principal theorem as in subsection 1.4.2.

Suppose Γ is the *-semi-group of non-negative integers n with addition as the "semi-group operation" and with the identity operation $n^*=n$ as the "*-opeartion;" then the "unit" element is the number 0.

Every representation of Γ is obviously of the form $\{A^n\}$ where A is a bounded self-adjoint operator.

We shall show that the sequence $\{A_n\}$(n=0,1,…,) visualized in Theorem 3.2, considered as a function of the variable element in the -semi-group , satisfies the conditions of the principal theorem. Condition (a) is obviously satisfied; as for the other two conditions, one proves them using the integral formula

$$A_n=\int_{-M-o}^{M}\lambda^n dB_\lambda$$

established in section 3, where $\{B_\lambda\}$ is a generalized spectral family on the interval [–M, M]. In fact, if $\{g_n\}$(n=0,1,…) is any sequence of elements in., which are almost all equal to 0, we have

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\left(A_{i+k}g_k,g_i\right)$$
$$=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\int_{-M-0}^{M}\lambda^{i+k}d\left(B_\lambda g_k,g_i\right)$$

$$\int_{-M-0}^{M}\left(B(d\lambda)g(\lambda),g(\lambda)\right)\geqq 0$$

Where we have set

$$g(\lambda)=\sum_{i=0}^{\infty}\lambda^i g_i$$

and where $B(\Delta)$ denotes the positive, additive interval function generated by B_λ, that is $B(\Delta)=B_b-B_a$ for $\Delta=(a,b)$. Furthermore, for r=0,1,…, we have that

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\left(A_{i+2r+k}g_k,g\right)$$
$$=\int_{-M-0}^{M}\lambda^{2r}(B\left(d(\lambda)g(\lambda),g(\lambda)\right))$$

$$\leqq M^{2r}\int_{-M-0}^{M}\left(B(d\lambda)g(\lambda),g(\lambda)\right)$$
$$=M^{2r}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\left(A_{i+k}g_k,g_i\right)$$

Thus, we see that conditions (b) and (c) are also satisfied, and one can then apply the principal theorem.

Furthermore, one can require that H be minimal in the sense that it be spanned by elements of the form $A^n\ T$ where T∈H and n=0,1,…; in this case, this structure $\{H,A,H\}$ is determined to within an isomorphism, and we have

$\|A\|\leqq M$

We observe that if $\{B_\lambda\}$ is a generalized spectral family on the interval [–M, M] (that is, $B_\lambda=0$ for $\lambda<-M$ and $B_\lambda=I$ for $\lambda\leqq M$), the operators

$$A_n=\int_{-M-0}^{M}\lambda^n dB_\lambda \qquad (n=0,1,\ldots) \qquad (1.3.3)$$

satisfy conditions (a_M) and (β). Conversely, if these conditions are satisfied, the sequence $\{A^n\}$ has an integral decomposition of the form (8) with $\{B_\lambda\}$ on [–M,M]. This clearly follows from Theorem 3.4 if we make use of the directly, in fact, the correspondence between the polynomials

$p(\lambda)=a_0+a_1\lambda+a_2\lambda^2+\cdots+a_n\lambda^n$

and the self-adjoint operators

$A(p)=a_0I+a_1A_1+a_2A_2+a_nA_n$

which is homogeneous, additive, and of positive type with respect to the interval $-M\leqq\lambda\leqq M$, can be extended, with preservation of these properties, to a vaster class of functions which comprises among others, the discontinuous functions

$$e_\mu(\lambda)=\begin{cases}1\; for\lambda\leqq\mu\\0\; for\;\lambda>\mu\end{cases}$$

and then we obtain representation (8) by setting $B\mu=A(e\mu)$.

We have only to repeat verbatim the line of the argument of one of the usual proofs of the spectral decomposition of a bounded self-adjoint operator A, letting A_n play the role of A^n. The only difference is that now the difference is that now the correspondence $p(\lambda)\to p(A)$ and its extension are no longer multiplicative and that consequently the relation $e_\mu^{\,2}(\lambda)\equiv e_\mu(\lambda)$ does not imply that $B_\mu^{\,2}$ is equal to B_μ and hence that B_μ is in general not a projection. According to Theorem 3.1, $\{B_\lambda\}$ is the projection of an ordinary spectral family $\{E_\lambda\}$, which one can choose in such a way that it is also on $[-M,M]$, and then (7) follows from (8) by setting

$$A=\int_{-M-0}^{M}\lambda dE_\lambda$$

We shall return to this theorem later and prove it as a corollary to the principal theorem.

2. If we replace condition (β) by less restrictive condition

$$A_0\leqq I, \tag{1.3.4}$$

Then, representation (7) of the sequence $\{A_n\}$ will still be possible, if only starting from $n=1$, everything reduces to showing that if the sequence

$$\{A_0,A_1,A_2,\ldots\}$$

satisfies conditions (a_M) and (β'), the sequence

$$\{I,A_1,A_2,\ldots,\}$$

satisfies condition (a_M). However, if $p(\lambda)=a_0+a_1\lambda+\cdots+a_n\lambda^n\geqq0$ in $[-M,M]$, we have in particular that $p(0)=a_0\geqq0$; since by assumption, $a_0A_0+a_1A_1+\cdots+a_nA_n\geqq0$ and $I-A\geqq0$, it follows that

$$a_0I+a_1A_1+\cdots+a_nA_n=a_0(I-A_0)+a_0A_0+a_1A_1+\cdots+a_nA_n\geqq0$$

One of the most interesting consequences of the representation

$$A_n=prA^n\ (n=1,2,\ldots)$$

Is the following. We have

$$(A_2T,T_0)=(PA^2T,T_0)=(A^2T,T)=\|AT_0\|^2$$
$$\geqq\|PAT_0\|^2=(APAT_0,T_0)=(PAPAT_0,T_0)=(A_1^{\,2}T_0,T_0)$$

for all $T\in\mathfrak{H}$, where equality holds if and only if

$$AT=PAT_0=A_1T_0.$$

If this case occurs for all $T\in\mathfrak{H}$, we have

$$A^2T_0=A(AT_0)=A(A_1T_0)=A_1(A_1T_0)=A_1^{\,2}T_0,$$
$$A^3T_0=A(A^2T_0)=A(A_1^{\,2}T_0)=A_1(A_1^{\,2}T_0)=A_1^{\,3}T_0,\text{ etc.}$$

And hence

$$A_nT_0=PA^nT_0=A_1^{\,n}T_0\ (n=1,2,\ldots,)$$

We have thus obtained the following result.

If the sequence A_0, A_1, A_2,… of bounded self-adjoint operators in the Hilbert space $\mathfrak{H}$ satisfies hypotheses (a_M) and (β'), then the inequality

$$A_1^{\,2}\leqq A_2 \tag{1.3.5}$$

holds, where equality occurs if and only if $A_n=A_1^{\,n}$ ($n=1,2,\ldots$). Inequality (9) is due to R.V Kadison who proved it differently and used it in his researches on algebraic invariants of operator algebras. Moreover, one can also omit hypotheses (β'), and then the following inequality

$$A_1^{\,2}\leqq\|A_0\|A_2 \tag{1.3.6}$$

is obtained; in fact, we have only to apply inequality (9) to the sequence $\{\|A_0\|^{-1}A_n\}$.

Normal Extensions

We proved previously in particular that every bounded linear operator T in the complex Hilbert space can be represented as the projection of a normal operator in an extension space. The question arises: Does T even have a normal extension N?

If a normal extension N of T exists, then a fortiori $T=pr\,N$, and consequently $T^*=prN$, from which it follows that

$$\|TT_0\|=\|NT_0\|=\|N^* T_0\|\geqq\|PN^*T_0\|=\|TT_0\|$$

for all $T\in\mathfrak{H}$. The inequality

$$\|TT_0\|\geqq\|T^* T_0\| \text{ (for all } T_0\in\mathfrak{H}) \qquad (1.4.1)$$

is therefore a necessary condition that T has a normal extension. However, it is easy to construct examples of operator T which do not satisfy this condition.

Other less simple necessary conditions are obtained in the following manner. Suppose $\{g_i\}$ ($i=0,1,\ldots$) is a sequence of elements in $\mathfrak{H}$ almost all of which (that is with perhaps the exception of a finite number of them) are equal to the element 0 in $\mathfrak{H}$. We then have

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(T^i g_j, T^j g_i\right)$$
$$=\sum_i\sum_j (N^i g_j, N^j g_i)=\sum_i\sum_j\left(N^{*j}N^i g_j g_i\right)$$

$$\sum_i\sum_j\left(N^i N^{*j} g_j g_i\right)$$
$$=\sum_i\sum_j\left(N^{*j}g_j, N^{*i}g_i\right)=\sum_i \|N^{*i}g_i\|^2\geqq 0,$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(T^{i+1}g_j, T^{j+1}g_i\right)=$$
$$\|\sum_i \left(N^*\right)^{i+1} g_i\|^2\leqq\|N\|^{*2}\|\sum_i N^* g_i\|^2$$

from which we see that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(T^i g_j, T^j g_i)\geqq 0 \qquad (1.3.8)$$

and

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(T^{i+1}g_j, T^{j+1}g_i\right)$$
$$\leqq C^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(T^i g_j, T^j, g_i\right) \qquad (1.3.9)$$

with constant $C>0$. These two inequalities are, therefore, necessary conditions that T has bounded normal extension. However, these conditions are also sufficient. Namely, the following theorem holds

Theorem 1.4.1:[8,9] Every bounded linear operator in the Hilbert space which satisfies conditions (16) and (17) has a bounded normal extension N in an extension space H. One can even require that H be minimal in the sense that it is spanned by the elements of the form

$N^{*k}\, T_0$ where $T_0\in\mathfrak{H}$ and $k=0,1,\ldots$; in this case, the structure $\{H,N,\mathfrak{H}\}$ is determined to within an isomorphism.

For the present, we shall concern ourselves with a remark connecting the problems on extensions.

Principal theorem

The following three propositions are equivalent for any two bounded linear operators, T in and T_0 in $H(\geqq\mathfrak{H})$:

a. $T\leqq\boldsymbol{T}_0$;
b. $T=pr\boldsymbol{T}$ and $T^*\,T_0=pr\boldsymbol{T}^*\,\boldsymbol{T}_0$
c. $T^{*i}\,T_0^{\,k}=pt\boldsymbol{T}^{*i}\,\boldsymbol{T}_0^{\,k}$ for i, $k=0,1,\ldots$

CONTRACTIONS IN HILBERT SPACE

1. Whereas the projections of bounded self-adjoint operators are also self-adjoint, the projections of unitary operators are already of a more general type. In order that $T=prU$, with U unitary, it is necessary that

$$\|TT_0\|=\|PUT_0\|\leqq\|UT_0\|=\|T_0\|$$

for all $T_0\in\mathfrak{H}$, that is, $\|T\|\leqq 1$, and hence the operator T must be a contraction. However, this condition is not only necessary but also sufficient.

Theorem 2.1:[9,10] Every contraction T in the Hilbert space $\mathfrak{H}$ can be represented in an extension space H as the projection of a unitary operator U onto $\mathfrak{H}$. The theorem and the following simple construction of U are due to Halmos. As in section 3, let us consider the product space $H=\mathfrak{H}\times\mathfrak{H}$ and the following operator of H;

$$U=\begin{pmatrix} T & S \\ -Z & T^* \end{pmatrix}$$

$$\text{where}\quad S=\left(I-TT^*\right)^{\frac{1}{2}},\ Z=\left(I-T^*T\right)^{\frac{1}{2}} \qquad (2.1)$$

The relation $T=prU$ is obvious. We shall show that U is unitary, or what amounts to the same thing, that $U^*\,U$ and UU^* are equal to the identity operator I in H. Since S and Z are self-adjoint, we have

$$U^*U = \begin{pmatrix} T^* & -Z \\ S & T \end{pmatrix}\begin{pmatrix} T & S \\ -Z & T^* \end{pmatrix} = \begin{pmatrix} T^*T + Z^2 & T^*S - ZT^* \\ ST - TZ & S^2 + TT^* \end{pmatrix}$$

$$UU^* = \begin{pmatrix} T & S \\ -Z & T^* \end{pmatrix}\begin{pmatrix} T^* & -Z \\ S & T \end{pmatrix} = \begin{pmatrix} TT^* + S^2 & -TZ + ST \\ -ZT^* + T^*S & Z^2 + T^*T \end{pmatrix}$$

Since $Z^2=I-T^*T$, $S^2=I-TT^*$, the diagonal elements of the product matrices are all equal to I. It remains to show that the other elements are equal to 0, that is, that

$$ST=TZ \qquad (2.2)$$

(the equation $T^*S=ZT^*$ follows from this by passing over to the adjoints of both members of (2.2)). However, we have

$$S^2 T=(I-TT^*)T=T-TT^*T=T(I-T^*T)=TZ^2,$$

from which it follows by complete induction that

$$S^{2n} T=TZ^{2n} \text{ for } n=0,1,2,\ldots$$

Then, we also have

$$p(S^2)T=Tp(T^2)$$

for every polynomial $p(\lambda)$. Since S and Z are the positive square roots of S^2 and Z^2 respectively, there exists a sequence of polynomials $p_n(\lambda)$ such that

$$p_n(S^2)\to S,\ p_n(Z^2)\to Z.$$

Now, (12) follows from the equation

$$p_n(S^2)T=Tp_n(Z^2)$$

by passing to the limit as $n\to\infty$. This completes the proof of the theorem

2. The relation between operators S in an extension space H of the space and their projections $T=prS$ onto $\mathfrak{H}$ is not multiplicative in general, that is, the equations $T_1=prS_1, T_2=prS_2$ do not in general imply $T_1 T_2=prS_1 S_2$. For example, if we consider the operator U constructed according to the formula (2.1), we have $prU^2=T^2-SZ$, which in general is not equal to T^2.

The question arises: Is it possible to find, in a suitable extension space, a unitary operator U such that the powers of the contraction T (which are themselves contractions) are at the same time equal to the projections onto $\mathfrak{H}$ of the corresponding powers of U?

If we are dealing with only a finite number of powers,

$$T, T^2, \ldots, T^k$$

then the problem can be solved in the affirmative in a rather simple manner suitably generalizing the immediately preceding construction. Let us consider the product space $H = \mathfrak{H}\times\ldots\times\mathfrak{H}$, with $k+1$ factors, whose elements are ordered $(k+1)$-tuples $\{T_1,\ldots,T_{k+1}\}$ of elements in $\mathfrak{H}$ and in which the vector operations and metric are defined in the usual way:

$$c\{T_1,\ldots,T_{k+1}\}=\{cT_1,\ldots,cT_{k+1}\},$$

$$\{T_1,\ldots,T_{k+1})\}+\{g_1,\ldots,g_{k+1}\}=\{T_1+g_1,\ldots,T_{k+1}+g_{k+1}\}$$

$$(\{T_1,\ldots,T_{k+1})\},\{g_1,\ldots,g_{k+1})\})=(T_1,g_1)+\cdots+(T_{k+1}),g_{k+1})$$

We embed $\mathfrak{H}$ in H as subspace of the latter by identifying the element T in with the element $\{T, 0,\ldots,0\}$ in H. The bounded linear operator $\boldsymbol{T}$ in H will be represented by matrices (T_{ij}) with $k+1$ rows and $k+1$ columns, all of whose elements T_{ij} are bounded linear operators in . We have $T=prT$ if and only if $T_{11}=T$.

Let us now consider the following operators in H:

$$U = \left.\begin{pmatrix} T & S & 0 & 0 & \ldots & 0 \\ 0 & 0 & -1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & -1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & -1 \\ -Z & T^* & 0 & \ldots & 0 & 0 \end{pmatrix}\right\} k+1$$

rows and columns

Where S and Z have the same meaning as in the foregoing construction. The operator U is unitary. This is proved in the same way as above, by a direct calculation of the matrices U^*U, UU^*. To prove the relations

$T^n = prU$ (n=1,2,…,k)

We must calculate the element in the matrix U^n having indices 1,2 and then note that the latter is equal to T^n for n=1,…,k. We shall even prove more namely that the first row in the matrix U^n (n=1,…,k) is the following:

$$\left(T^n, T^{n-1}S, -T^{n-2}S^2, T^{n-3}S^3, \ldots, (-1)^{n-1}S^n, \overbrace{0 \ldots 0}^{k-n}\right)$$

This proposition is obvious for n=1 and we prove it true for n+1, assuming it true for $n(n \leqq k-1)$ by calculating the matrix U^{n+1} as the matrix product U^n. U we have thus proved the following theorem.

Theorem 2.2:[10,11] If T is a contraction in the Hilbert space $\mathfrak{H}$, then there exists a unitary operator U in an extension space H such that

$T^n = prU^n$ n=0,1,…,k

(the case n=0 is trivial), for every given natural number k. The product space $\mathfrak{H} \times \ldots \times \mathfrak{H}$ with k+1 factors can be written for H.

3. It is important in the above construction that k is a finite number; however, the theorem is also true for $k=\infty$.

Theorem 2.3:[12,13] If T is a contraction in the Hilbert space $\mathfrak{H}$, then there exists a unitary operator U of an extension space H such that the relation

$T^n = prU^n$

is valid for n=0,1,2,… Furthermore, one can require that the space H be minimal in the sense that it is spanned by the elements of the form U^nT where $T \in \mathfrak{H}$ and n=0, ±1, ±2,…, in this case, the structure $\{H, U, \mathfrak{H}\}$ is determined to within an isomorphism.

An analogous theorem is true for semi-groups and one-parameter semi-groups of contractions, that is, for families $\{T_1\}$ of contractions (where $0 \leqq t < \infty$) or $-\infty \leqq t < \infty$, according to the case at hand) such that

$$T_0 = I,\ T_{t_1}T_{t_2} = T_{t_1+t_2},$$

And for which one assumes further that T_t depends strongly or weakly continuously on t; weak continuity means that (T, f, g) is a continuous numerical-valued function of t for every pair T, g of elements $\mathfrak{H}$.

Theorem 2.4:[13,14] If $\{T_t\}_{t \geqq 0}$ is a weakly continuous one-parameter semi-group of contractions in the Hilbert space $\mathfrak{H}$, then there exists a one-parameter group $\{U_t\}_{-\infty<t<\infty}$ of unitary operator in an extension space H, such that

$T_t = prU_t$ for $t \geqq 0$

Furthermore, one can require that the space H be minimal in the sense that it is spanned by elements of the form U_tf, where $f \in \mathfrak{H}$ and $-\infty<t<\infty$; in this case, U_t is strongly continuous and the structure $\{H, U_t, \mathfrak{H}\}_{-\infty<t<\infty}$ is determined to within an isomorphism.

These two theorems can be generalized to discrete or continuous semi-groups with several generators. We shall formulate only the following generalization of Theorem 2.3.

Theorem 2.5:[14,15] Suppose $\{T^{(\rho)}\}_{\rho \in R}$ is a system of pairwise doubly permutable contraction in the Hilbert space $\mathfrak{H}$. There exists in an extension space H, a system $\{U^{(\rho)}\}_{\rho \in R}$ of pair-wise unitary transformations such that

$$\prod_{i=1}^{r} \left[T^{(\rho_i)}\right]^{n_i} = pr\prod_{i=1}^{r} \left[U^{(\rho_i)}\right]^{n_i}$$

for arbitrary $\rho_i \in R$ and integers n_i, provided the factor $\left[T^{(\rho_i)}\right]^{n_i}$ is replaced by $\left[T^{(\rho_i)^*}\right]^{-n_i}$ when $n_i<0$. Moreover, one can require that the space H be minimal in the sense that it be spanned by the elements of the form $\prod_{i=1}^{r} \left[U^{(\rho_i)}\right]^{n_i} T$ where $T \in$.; in this case, the structure $\{H, U^{(\rho)}, \mathfrak{H}\}_{\rho \in R}$ is determined to within an isomorphism.

APPLICATION TO ELASTICITY, DEFORMATION, AND STRESS

Deformation and Stress

In this section, we discuss how the position of each particle may be specified at each instant and we introduce certain measures of the change of shape and size of infinitesimal elements of the material. These measures are known strains and they are used later in the derivation of the equations of elasticity. We also consider the nature of the forces acting on arbitrary portions of the body and this leads us into the concept of stress.[15,16]

Motion, material, and spatial coordinates

We[16,17] wish to discuss the mechanics of bodies composed of various materials. We idealize the concept of a body by supposing that it is composed of a set of particles such that, at each instant of time t, each particle of a set is assigned to a unique point of a closed region l_t of three-dimensional Euclidean space and that each point of l_t is occupied by just one particle. We call l_t the configuration of the body at time t.

To describe the motion of the body, that is, to specify the position of each particle at each instant, we require some convenient method of labeling the particles. To do this, we select one particle configuration l and call this the reference configuration. The set of coordinates (X_1, X_2, X_3), or position vector X, referred to fixed Cartesian axes of a point of l uniquely determines a particle of the body and may be regarded as a label by which the particle can be identified for all time. We often refer to such a particle as the particle X. In choosing l, we are not restricted to those configurations occupied by the body during its actual motion, although it is often convenient to take l to be the configuration l_0 occupied by the body at some instant which is taken as the origin of the time scale t. The motion of the body may now be described by specifying the position x of the particle X at time t in the form of an equation

$$x=x(X,t) \tag{3.1}$$

[Figure 1] or in component form,

$$x_1=x_1\,(X_1,X_2,X_3,t),\ x_2=x_2\,(X_1,X_2,X_3,t),\ x_3=x_3\,(X_1,X_2,X_3,t) \tag{3.2}$$

Moreover, we assume that the functions x_1,x_2 and x_3 are differentiable with respect to X_1,X_2,X_3 and t as many as required. Sometimes we wish to consider only two configurations of the body, an initial configuration and a final configuration. We refer to the mapping from the initial to the final

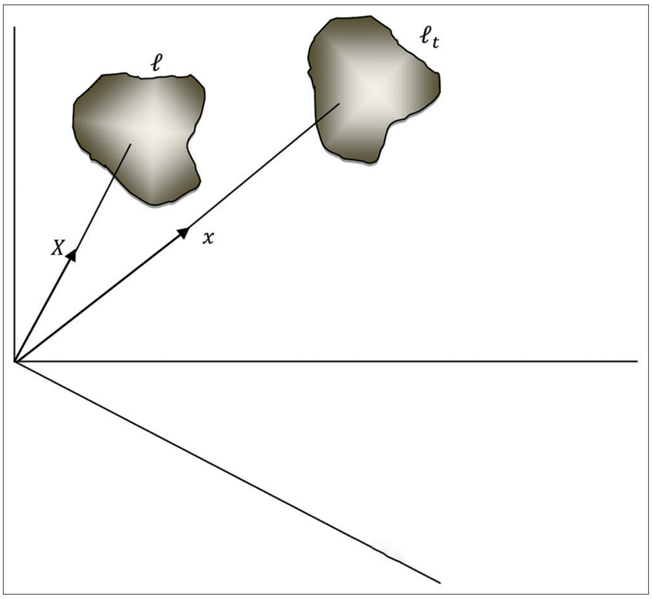

Figure 1: Motion, material, and spatial coordinates

configuration as a deformation of the body, which is either contraction or expansion. The motion of the body may be regarded as a one-parameter sequence of deformations.

We assume that the Jacobian

$$J=\det\left(\frac{\partial \chi_i}{\partial X_A}\right),\quad i,A=1,2,3 \tag{3.3}$$

Exists at each point of l_t, and that

$$J>0 \tag{3.4}$$

The physical significance of these assumptions[16,17] is that the material of the body cannot penetrate itself and that material occupying a finite non-zero volume in l cannot be compressed to a point or expanded to infinite volume during the motion. Mathematically, (3.4) implies that (3.1) has the unique inverse

$$X=\chi^{-1}\,(x,t) \tag{3.5}$$

Now, at the current time t, the position of a typical particle P is given by its Cartesian coordinates (x_1, x_2, and x_3), but as mentioned above, P continues to be identified by the coordinates (X_1, X_2, and X_3) which denoted its position in l. The coordinates (X_1, X_2, and X_3) are known as material (or Lagrangian) coordinates since distinct sets of these coordinates refer to distinct material particles. The coordinates (x_1, x_2, and x_3) are known as spatial (or Eulerian) coordinates since distinct sets refer to distinct points of the space. The values of x given by equation (3.1) for a fixed value of X are those points of space occupied by the particle X during the motion. Conversely, the values of X given by equation (3.5) for a fixed value of x identify the particles X passing through the point x during the motion.

From now on, when the upper or lower case letters are used as suffixes, they are understood generally to range over 1, 2, and 3. Usually, the upper case suffixes refer to material coordinates, the lower case to spatial and repetition of any suffix refers to summation over the range. For example, we write x_i for (x_1,x_2,x_3),X_A for (X_1,X_2,X_3) and $x_i\ x_i$ denotes $x_1^2+x_2^2+x_3^2$.

When a quantity is defined at each point of the body at each instant of time, we may express this quantity as a function of X_A and t or of x_i and t. If X_A and t are regarded as the independent variables then the function is said to be a material description of the quantity; if x_i and t are used then the corresponding function is said to be a spatial description. One description is easily transformed into the other using (3.1) or (3.5). The material description spatial description $\psi(X,t)$ has a corresponding spatial description $\psi(x,t)$ related by

$$\psi(\chi^{-1}(x,t),t)=\psi(x,t) \tag{3.6}$$

or

$$\psi(\chi(X,t),t)=\psi(X,t) \tag{3.7}$$

To avoid the use of a cumbersome notation and the introduction of a large number of symbols, we[17,18] usually omit explicit mention of the independent variables and also use a common symbol for a particular quantity and regard it as denoting sometimes a function of X_A and t and sometimes the associated function of x_i and t. The following convention for partial differentiation should avoid any confusion.

Let u be the common symbol used to represent a quantity with the material description ψ and spatial description ψ (these may be scalar-, vector-, or tensor-valued functions) as related by (3.6) and (3.7). We adopt the following notation for the various partial derivatives:

$$u_K=\frac{\partial\psi}{\partial X_K}(X,t),\quad \frac{Du}{Dt}=\frac{\partial\psi}{\partial t}(X\ t) \tag{6.8}$$

$$u_i=\frac{\partial\psi}{\partial x_i}(x,t),\qquad \frac{\partial u}{\partial t}=\frac{\partial\psi}{\partial t}(x,t) \tag{6.9}$$

The material time derivative

Suppose[18,19] that a certain quantity is defined over the body and we wish to know its time rate of the change as would be recorded at a given particle X during the motion. This means that we must calculate the partial derivative with respect to time of the material description ψ of the quantity keeping X fixed. In other words, we calculate $\partial\psi(X,t)/\partial t$. This quantity is known as a material time derivative. We may also calculate the material time derivative from the spatial description ψ. Using the chain rule of partial differentiation, we see from (3.7) that

$$\frac{\partial\psi}{\partial t}(X,t)=\frac{\partial\psi}{\partial t}(x,t)+\frac{\partial\chi_i}{\partial t}(X,t)\frac{\partial\psi}{\partial x_i}(x,t) \tag{3.10}$$

remembering of course that repeated suffixes imply summation over 1, 2, and 3. Consider now a given particle X_0. Its position in the space at time t is

$$x=\chi(X_0,t)$$

and so its velocity and acceleration are

$$\frac{d\chi}{dt}(X_0,t)\ \text{and}\ \frac{d^2\chi}{dt^2}(X_0,t)$$

respectively. We, therefore, define the velocity field for the particles of the body to be material time derivative $\partial\chi(X,t)/\partial t$, and use the common symbol v to denote its material or spatial description.

$$v=\frac{\partial\chi}{\partial t}(X,t)=\frac{Dx}{Dt} \tag{3.11}$$

Likewise, we define the acceleration field f to be the material time derivative of v

$$f=\frac{Dv}{Dt} \tag{3.12}$$

Moreover, in view of (6.10), the material time derivative of has the equivalent forms

$$\frac{Du}{Dt}=\frac{\partial u}{\partial t}+v_i u_i \tag{3.13}$$

In particular, the acceleration (3.12) may be written as

$$f=\frac{Dv}{Dt}=\frac{\partial v}{\partial t}+\left(v.\nabla\right)v \qquad (3.14)$$

where the operator ∇ is defined relative to the coordinates x_i, that is

$$\nabla=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right) \qquad (3.15)$$

In suffix, notation (6.14) becomes

$$f_i=\frac{Dv_i}{Dt}=\frac{\partial v_i}{\partial t}+v_j v_{i,j} \qquad (3.16)$$

The deformation-gradient tensor

We[19,20] have discussed how the motion of a body may be described. In this section, we analyze the deformation of infinitesimal elements of the body which results from this motion. Suppose that l coincides with the initial configuration l_0, and that two neighboring particles P and Q have positions X and $X+dX$ in l. Then, at time t, their positions in l_t are and $x+dx$. Where

$$x=\chi(X,t),\ x+dx=\chi(X+dX,t) \qquad (3.17)$$

and the components of the total differential dx are given in terms of the components of dX and the partial derivatives of χ by

$$dx_i=\frac{\partial\chi_i}{\partial X_A}\left(X,t\right)dX_A=x_{i,A}dX_A \qquad (3.18)$$

The quantities $x_{i,A}$ are known as the deformation gradients. They are the components of a second-order tensor known as the deformation-gradient tensor which we denote by F.

Strain tensors

Denoting[20,21] the deformation gradients $x_{i,A}$ by $F_{i,A}$, equation (3.18) may be written

$$dxi=F_{iA}\,dX_A \qquad (3.19)$$

In view of our assumption (3.4), the tensor F is non-singular and so permits the unique decompositions

$$F=RU,\ F=VR \qquad (3.20)$$

Where U and V are positive-definite symmetric tensors and R is proper orthogonal. We note that a proper orthogonal tensor R has the properties

$$R^T R=RR^T=I,\ \det R=1 \qquad (3.21)$$

where R^T denotes the transpose of R, and I denotes the unit tensor. A positive-definite tensor U has the property

$$x_i\,U_{ij}\,x_j>0 \qquad (3.22)$$

For all non-null vectors x. To see the physical significance of the decomposition (3.20), we first write (3.19) in the form

$$dx_i=R_{iK}\,U_{KL}\,dX_L \qquad (3.23)$$

or equivalently,

$$dx_i=R_{iK}\,dy_K,\ dy_K=U_{KL}\,dX_L \qquad (3.24)$$

In other words, the deformation of line elements dX into dx, caused by the motion, may split into two parts. Since U is a positive-definite symmetric tensor, there exists a set of axes, known as principal axes, referred to which U is diagonal; and the diagonal components are the positive principal values U_1, U_2, and U_3 of U. Equation (3.24), referred to these axes, becomes

$$dy_1=U_1\,dX_1,\ dy_2=U_2\,dX_2,\ dy_3=U_3\,dX_3 \qquad (3.25)$$

In the deformation represented by equations (3.25), the *ith* component of each line element is increased or diminished in magnitude according as $U_i>1$ or $U_i<1$. This part of the deformation.

Therefore,[3,21] amounts to a simple stretching or compression in three mutually perpendicular directions. (of course if $U_i=1$ the corresponding component of the line element is unchanged). The values of U_i are known as the principal stretches. Equation (3.24) describes a rigid body rotation of the line elements dy to dx. Hence, the line elements dX may be thought of as being first translated from X to x, then stretched by the tensor U as described above, and finally rotated as a rigid body in a manner determined by R [Figure 2].

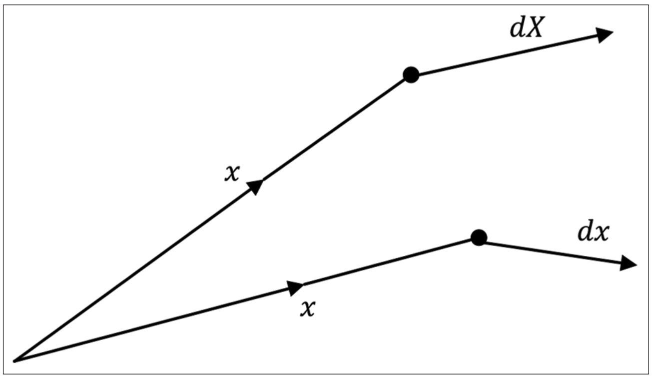

Figure 2: Strain tensors

The decomposition (3.20) may be interpreted in a similar way, although it should be noted that in this case, the rotation comes before the stretching. The tensors U and V are known as the right and left stretching tensors, respectively.

Although the decomposition (3.20) provides useful measures of the local stretching of an element of the body as distinct from its rigid body rotation, the calculation of the tensors U and V for any but the simplest deformations can be tedious. For this reason, we define two more convenient measures of the stretching part of the deformation. We define the right and left Cauchy-Green strain tensors

$$C=F^TF,\ B=FF^T \qquad (3.26)$$

respectively. Clearly, C and B are symmetric second-order tensors. The tensor C is easily related to U since using (3.20) and (3.21)

$$C=U^T R^T RU=U^T U=U^2 \qquad (3.27)$$

Similarly, we can show that

$$B=V^2 \qquad (3.28)$$

As can be seen from the definitions (3.26), when F has been found, the tensors B and C are easily calculated by matrix multiplication; and in principle, U and V can be determined as the unique positive-definite square roots of C and B are diagonal. In such cases U and V can be found easily.

Example 3.1:[21,22]

Find the tensors F, C, B, U, V, and R for the deformation

$$x_1=X_1, x_2=X_2-\alpha X_3, x_3=X_3+\alpha X_2 \qquad (3.29)$$

where $\alpha(>0)$ is a constant and interpret the deformation as a sequence of stretches and a rotation.

For this deformation

$$F=\begin{pmatrix}1&0&0\\0&1&-\alpha\\0&\alpha&1\end{pmatrix},\quad J=1+\alpha^2>0 \qquad (6.30)$$

Hence,

$$C=F^TF=\begin{pmatrix}1&0&0\\0&1&\alpha\\0&-\alpha&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&-\alpha\\0&\alpha&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1+\alpha^2&0\\0&0&1+\alpha^2\end{pmatrix} \qquad (3.31)$$

and therefore

$$U=\begin{pmatrix}1&0&0\\0&\left(1+\alpha^2\right)^{\frac{1}{2}}&0\\0&0&\left(1+\alpha^2\right)^{\frac{1}{2}}\end{pmatrix} \qquad (3.32)$$

It can easily be shown likewise that $B=C$ and $V=U$. We may calculate R from the relation $R=FU^{-1}$. Thus,

$$R=\begin{pmatrix}1&0&0\\0&1&-\alpha\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&\left(1+\alpha^2\right)^{\frac{1}{2}}&0\\0&0&\left(1+\alpha^2\right)^{\frac{1}{2}}\end{pmatrix}$$

$$=\begin{pmatrix}1&0&0\\0&\left(1+\alpha^2\right)^{\frac{1}{2}}&-\alpha\left(1+\alpha^2\right)^{-\frac{1}{2}}\\0&\alpha\left(1+\alpha^2\right)^{-\frac{1}{2}}&\left(1+\alpha^2\right)^{-\frac{1}{2}}\end{pmatrix} \qquad (3.33)$$

Now, let $\alpha=\tan\theta\left(0<\theta<\frac{1}{2}\pi\right)$, then

$$R=\begin{pmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{pmatrix}$$

which represents a rotation through an angle $-\theta$ about the 1-axis, using the usual corkscrew convention for the sign of the angle. Thus, the deformation may be accomplished by first performing stretches of magnitudes $\left(1+\alpha^2\right)^{\frac{1}{2}}$ in the 2- and 3-directions and then a rotation about the 1-axis. Since in this example $B=C$ and $V=U$, these operations may be reversed in order.

If a portion of the body moves in such a manner that the distances between every pair of particles remain constant that portion is said to move as a rigid body. For such a motion, no stretching of line elements occurs and so at each particle of the given portion,

$$B=C=U=V=I,\ F=R \tag{3.34}$$

In general, of course, the motion of the body does produce changes in the lengths of line elements and an analysis of these length changes leads us to an alternative interpretation of C and B. Suppose that dL and dl denote the lengths of the vector line elements dX and dx, respectively. Then, using (3.18) and (3.26) and the Kronecker delta (defined by $\delta_{KL}=0, K\neq L, K=L$),

$$(dl)^2-(dL)^2=dx_i\, dx_i-dX_K\, dX_K$$

$$=x_{i,K}\, x_{i,L}\, dX_K\, dX_L-dX_K\, dX_K$$

$$=(C_{KL}-\delta_{KL})dX_{KL}\, dX_L \tag{6.35}$$

and so the tensor C enables us to calculate the difference between the squared elements of length in the reference and current configurations. Alternatively, if we define the inverse deformation gradients using (3.5) as

$$X_{K,i}=\frac{\partial}{\partial x_i}\chi_K^{-1}\left(x,t\right) \tag{3.38}$$

then, since $X_{K,i}\ x_{i,A}=\delta_{KA}$ by chain rule of partial differentiation, it follows from (6.18) that

$$dX_K=X_{K,I}\, dx_i \tag{3.39}$$

Hence, we may write

$$(dl)^2-(dL)^2=dx_i\, dx_i-X_{K,i}\, X_{K,j}\, dx_i\, dx_i$$

It can easily be verified, using (3.26) and the result $(F^T)^{-1}=(F^{-1})^T$, that

$$(F^{-1})\ T\ F^{-1}=B^{-1},\ X_{K,I}\, X_{K,j}=B_{ij}^{-1} \tag{3.40}$$

and therefore

$$(dl)^2-(dL)^2=(\delta_{ij}-B_{ij}^{-1})dx_i\, dx_j \tag{3.41}$$

The tensor B also provides us with a means of calculating the same difference of squared elements of length. As we have already noted, B and C are second-order symmetric tensors. Their principal axes and principal values are real and may be found in the usual manner (Spencer (1980) Sections 2.3 and 9.3). The characteristic equation for C is

$$\det\,(C_{KL}-\lambda\delta_{KL})=0$$

that is,

$$\lambda^3-I_1\,\lambda^2+I_2\,\lambda-I_3=0 \tag{3.42}$$

where

$$I_1=C_{KK}=tr\ C$$

$$I_2=\frac{1}{2}\left(C_{KK}C_{LL}-C_{KL}C_{KL}\right)=\frac{1}{2}\left(tr\,C\right)^2-\frac{1}{2}tr\,C^2 \tag{3.43}$$

$I_3=\det C$

and tr denotes the trace. The quantities I_1, I_2, and I_3 are known as the principal invariants of C.

We also note here a useful physical interpretation of I_3. In view of the definitions (3.3) and (6.26)

$$I_3=\det\ C=(\det\ F)^2=J^2 \tag{3.44}$$

and if a given set of particles occupies an element of volume dV_0 in l and dV in l_t then using (6.21),

$$J=dV/dV_0 \tag{3.45}$$

Thus, recalling (6.4)

$$dV/dV_0=\sqrt{I_3} \tag{3.46}$$

If no volume change occurs during the deformation, the deformation is said to be isochronic and

$$J=1,\ I_3=1 \tag{3.47}$$

The strain invariants are also of fundamental importance in the constitutive theory of elasticity and we also find the following relation useful:

$$I_2=I_3\, tr(B^{-1}) \tag{6.48}$$

To prove this, we first note that from the Cayley-Hamilton theorem (Spencer (1980) Section 2.4), a matrix satisfies its own characteristic equation. Since the principal invariants of B are identical to those of C, B must satisfy the equation

$$B^3-I_1B^2+I_2B-I_3I=0 \tag{3.49}$$

Now, B is non-singular, so multiplying (6.49) by B^{-1}, we find that

$$B^2=I_1B-I_2I+I_3B^{-1}$$

Taking the trace of this equation, we have

$$tr\left(B^2\right)=I_1^2-3I_2+I_3tr\left(B^{-1}\right) \tag{3.50}$$

And using (3.43), (3.50) reduces to (3.48).

Homogeneous deformation

A deformation of the form,

$$x_i=A_{iK}X_K+a_i \tag{3.51}$$

in which A and a are constants, is known as a homogeneous deformation. Clearly, $F=A$ and $J=\det A$. Particularly, simple examples of such deformations are given below

i. Dilatation

Consider the deformation[21,22]

$$x_1=\alpha X_1,\ x_2=\alpha X_2,\ x_3=\alpha X_3 \tag{3.52}$$

where α is a constant, then

$$F=\alpha I,\ B=C=\alpha^2 I, J=\alpha^3 \tag{3.53}$$

and so, to satisfy (3.4), we must have $\alpha>0$. The strain invariants (3.41) are easily seen to be

$$I_1=3\alpha^2,\ I_2=3\alpha^4,\ I_3=\alpha^6 \tag{3.54}$$

In view of (3.45), we see that if $\alpha>1$ the deformation represents an expansion; if $\alpha<1$ the deformation becomes a contraction.

ii. Simple extension with lateral extension or contraction

Suppose that[21,22]

$$x_1=\alpha X_1,\ x_2=\beta X_2,\ x_3=\beta X_3 \tag{3.55}$$

Then,

$$F=\begin{pmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \beta\end{pmatrix},\quad J=\alpha\beta^2 \tag{3.56}$$

and so $\alpha>0$. If $\alpha>1$, the deformation is a uniform extension in the 1-direction. If $\beta>0$, the diagonal terms of F are all positive so that $U=F$ and $R=1$; β measures the lateral extension ($\beta>1$), or contraction ($\beta<1$), in the 2,3-plane.

If $\beta<0$ then

$$U=\begin{pmatrix}\alpha & 0 & 0\\ 0 & -\beta & 0\\ 0 & 0 & -\beta\end{pmatrix},\quad R=\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}$$

In this case $-\beta$ measures the associated lateral extension or contraction and a rotation through an angle π about the 1-axis is included in the deformation.

If the material is incompressible only isochoric deformation is possible in which case

$$J=\alpha\beta^2=1 \tag{3.57}$$

This means that $|\beta|$ is less than or greater than unity according as to whether α is greater than or less than unity. In other words, an extension in the 1-direction produces a contraction in the lateral directions and vice versa. The strain tensors are found to be

$$B=C=\begin{pmatrix}\alpha^2 & 0 & 0\\ 0 & \beta^2 & 0\\ 0 & 0 & \beta^2\end{pmatrix} \tag{3.58}$$

and the invariants are

$$I_1=\alpha^2+2\beta^2,\ I_2=2\alpha^2\beta^2+\beta^4,\ I_3=\alpha^2\beta^4 \tag{3.59}$$

Example 3.2:[22,23] The previous two deformations are special cases of

$x_1=\lambda_1 X_1, x_2=\lambda_2 X_2, x_3=\lambda_3 X_3$ (3.60)

where λ_i (i=1, 2, and 3) are constants. Show that, for the deformation (6.60) to satisfy J>0, at least one of the λ_i has to be positive. Interpret the deformation geometrically in the case when all the λ_i is positive and show that in all cases, the principal invariants are

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2,$$
$$I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2,$$
$$I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 \quad (3.61)$$

iii. Simple Shear

Consider the deformation

$x_1=X_1+\kappa X_2, x_2=X_2, x_3=X_3$ (3.62)

where κ is a constant. The particles move only in the 1-direction, and their displacement is proportional to their 2-coordinate. This deformation is known as a simple shear. Plane parallel to X_1=0 are rotated about an axis parallel to the 3-axis through an angle $\theta=tan^{-1}\kappa$, known as the angle of shear. The sense of the rotation is indicated in Figure 3. The planes X_3 = constant are called shearing planes; and lines parallel to X_3-axis are known axes of shear. The deformation-gradient tensor is

$$F = \begin{pmatrix} 1 & \kappa & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.63)$$

and the Cauchy-Green strain tensors are

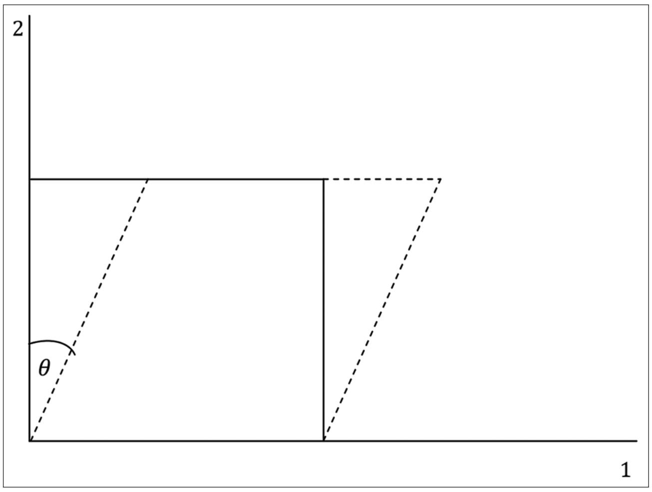

Figure 3: Simple Shear

$$C = F^T F = \begin{pmatrix} 1 & \kappa & 0 \\ \kappa & 1+\kappa^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$B = FF^T = \begin{pmatrix} 1+\kappa^2 & \kappa & 0 \\ \kappa & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.64)$$

The tensor B^{-1} may be found using the formula

B^{-1}=adj B/det B (3.65)

where adj B denotes the adjoint matrix of B. Thus

$$B^{-1} = \begin{pmatrix} 1 & -\kappa & 0 \\ -\kappa & 1+\kappa^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.66)$$

The strain invariants are

$I_1=3+\kappa^2, I_2=3+\kappa^2, I_3=1$ (3.67)

Non-homogeneous deformations

Deformations which are not of the form (6.51) are referred to as non-homogeneous deformations. We now discuss two such deformations which may be applied to either a solid or hollow circular cylinder. In each case, we take our coordinate system such that the X_3-axis coincides with the axis of the cylinder and the base lies in the plane X_3=0.

i. Simple Torsion

Consider the deformation in which each cross-section remains in its original plane but is rotated through an angle τX_3 about the 3-axis, where τ is a constant called the twist per unit length [Figure 4]. This deformation is referred to as simple torsion. Since each cross-section remains in its original plane,

$x_3=X_3$ (3.68)

To find the remaining equations which specify this deformation, we consider the typical cross-section, as shown in Figure 5, which is at a distance X_3 from the base of the cylinder. If P is the particle whose initial coordinates are (X_1,X_2), then we may write

$X_1=R\cos\alpha, X_2=R\sin\alpha$ (3.69)

where $R = \left(X_1^2 + X_2^2\right)^{\frac{1}{2}}$. After the deformation, this particle occupies the point p with coordinates (x_1,x_2) where from the figure, it follows that

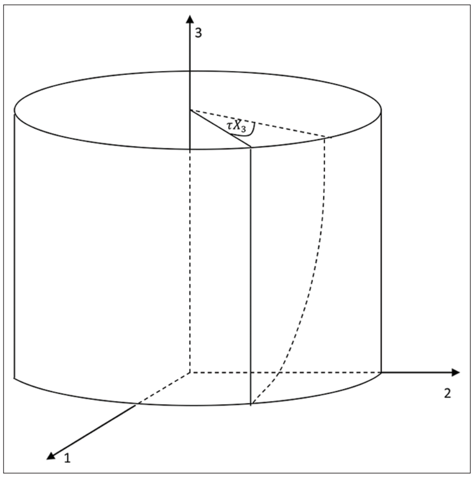

Figure 4: Simple Torsion

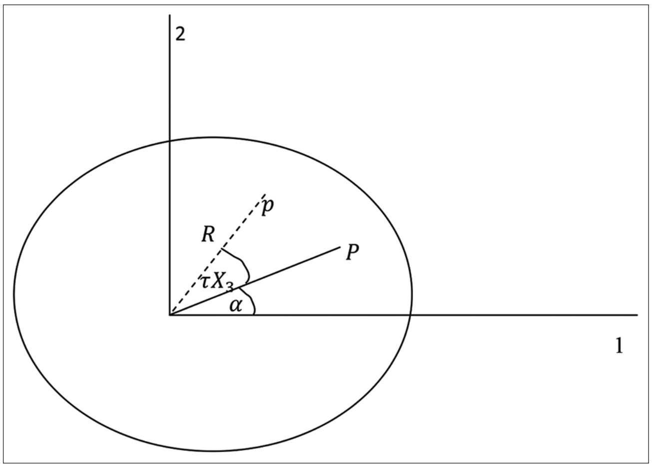

Figure 5: Torsion, extension and inflation

$$x_1=R\cos(\tau X_3+\alpha),\ x_2=R\sin(\tau X_3+\alpha) \tag{3.70}$$

Expanding the sine and cosine functions and using (6.69), we obtain

$$x_1=cX_1-sX_2, x_2=sX_1+xX_2 \tag{3.71}$$

where c=cos τX_3 and s=sin τX_3. For the deformation specified by (3.68) and (3.72), the deformation gradient is given by

$$F=\begin{pmatrix} c & -s & -\tau\left(sX_1+cX_2\right) \\ s & c & \tau(cX_1-sX_2) \\ 0 & 0 & 1 \end{pmatrix} \tag{3.72}$$

so that J=1 and the deformation is isochoric. Further,

$$B=\begin{pmatrix} 1+\tau^2(sX_1+cX_2) & -\tau^2(sX_1+cX_2)(cX_1-sX_2) & -\tau(sX_1+cX_2) \\ -\tau^2(sX_1+cX_2)(cX_1-sX_2) & 1+\tau^2\left(cX_1-sX_2\right)^2 & \tau(cX_1-sX_2) \\ -\tau(sX_1+cX_2) & \tau(cX_1-sX_2) & 1 \end{pmatrix}$$

(3.73)

Since J=1, det B=J^2=1 and using (6.65), it follows that

$$B^{-1}=\begin{pmatrix} 1 & 0 & \tau(sX_1+cX_2) \\ 0 & 1 & -\tau(cX_1-sX_2) \\ \tau(sX_1+cX_2) & -\tau(cX_1-sX_2) & 1+\tau^2(X_1^2+X_2^2) \end{pmatrix}$$

3.74)

Using (6.48), (6.72) and (6.73), we obtain

$$I_1=I_2=3+\tau^2 R^2 \tag{3.75}$$

ii. Torsion, extension and inflation

Finally, we[22,23] discuss the deformation which corresponds to simple extension along the axis of the cylinder, followed by simple torsion about its axis with twist τ per unit length. As a result of a uniform extension along the axis, the particle X is displaced to X', where

$$X_1'=\beta X_1,\ X_2'=\beta X_2,\ X_3'=\alpha X_3 \tag{3.76}$$

and here, we take α>0,β>0. If we now apply simple torsion to the extended cylinder, using (3.68) and (3.71), the final position x of the particle X is given by

$$x_1=X_1'\cos\left(\tau X_3'\right)-X_2'\sin\left(\tau X_3'\right),$$

$$x_2=X_1'\sin\left(\tau X_3'\right)+X_2'\cos\left(\tau X_3'\right)$$

$$x_3=X_3' \tag{3.77}$$

Combining (6.76) and (6.77), we obtain the deformation

$$x_1=\beta\{X_1\cos(\alpha\tau X_3)-X_2\sin(\alpha\tau X_3)\},$$

$$x_2=\beta\{X_1\sin(\alpha\tau X_3)+X_2\cos(\alpha\tau X_3)\}, x_3=\alpha X_3 \tag{3.78}$$

As a result of this deformation, the length of the cylinder increases or decreases depending on whether α>1 or α<1. Also, since

$$x_1^2+x_2^2=\beta^2\left(X_1^2+X_2^2\right) \tag{3.79}$$

The radius of the cylinder increases if β>1 and decreases if β<1. The deformation is usually referred to as torsion, extension, and inflation. The deformation gradient is given by

$$F=\begin{pmatrix}\beta c & -\beta s & -\alpha\tau\beta(sX_1+cX_2)\\ \beta s & \beta c & \alpha\tau\beta\left(cX_1-sX_2\right)\\ 0 & 0 & \alpha\end{pmatrix}\qquad(3.80)$$

where $s=\sin\alpha\tau X_3$ and $c=\cos\alpha\tau X_3$, so that $J=\alpha\beta^2$. If the material is incompressible, only isochoric deformations are possible, in which case

$$\beta=\alpha^{-\frac{1}{2}}$$

Then

$$B=\begin{pmatrix}\alpha^{-1}+\alpha\tau^2\left(sX_1+cX_2\right)^2 & -\alpha\tau^2\left(sX_1+cX_2\right)\left(cX_1-sX_2\right) & -\alpha^{\frac{3}{2}}\tau\left(sX_1+cX_2\right)\\ -\alpha\tau^2\left(cX_1-sX_2\right)\left(sX_1+cX_2\right) & \alpha^{-1}+\alpha\tau^2\left(cX_1-sX_2\right)^2 & \alpha^{\frac{3}{2}}\tau\left(cX_1-sX_2\right)\\ -\alpha^{\frac{3}{2}}\tau\left(sX_1+cX_2\right) & \alpha^{\frac{3}{2}}\tau\left(cX_1-sX_2\right) & \alpha^2\end{pmatrix}$$

(3.81)

Since for isochoric deformation $J=1$, $\det B=1$ and

$$B^{-1}=\begin{pmatrix}\alpha & 0 & \alpha^{\frac{1}{2}}\tau(sX_1+cX_2)\\ 0 & \alpha & -\alpha^{\frac{1}{2}}\tau\left(cX_1-sX_2\right)\\ \alpha^{\frac{1}{2}}\tau(sX_1+cX_2) & -\alpha^{\frac{1}{2}}\tau\left(cX_1-sX_2\right) & \alpha^{-2}\{1+\alpha^2\tau^2(X_1^2\left(X_1^2+X_2^2\right)\end{pmatrix}$$

(3.82)

Also $I_3=1$ and from (3.81) and (3.82) using (3.48) it follows that

$$I_1=\alpha^2+2\alpha^{-1}\ \alpha\tau^2 R^2,\ I_2=2\alpha+\alpha^{-2}+\tau^2 R^2 \qquad (3.83)$$

Exact Solutions for Problems with Boundary Conditions

In this section, we investigate the possibility of finding exact solutions of the equations without restriction on the form of the strain-energy function except for that imposed in some cases by the incompressibility condition.

Basic equations, boundary conditions

Restricting our attention to bodies maintained in equilibrium, the remaining equations which have to be satisfied for a compressible material are

$$\rho J=\rho_0 \qquad (3.2.1)$$

and

$$T_{ij,j}+\rho b_i=0 \qquad (3.2.2)$$

where

$$T_{ij}=\chi_0\delta_{ij}+\chi_1 B_{ij}+\chi_{-1}B_{ij}^{-1} \qquad (3.2.3)$$

and χ_0, χ_1, and χ_{-1} are functions of I_1, I_2, and I_3.

One way to maintain a body in equilibrium is to apply suitable surface tractions on its boundary. This type of problem gives rise to traction boundary conditions involving the stress vector. Let us consider a few typical examples. When a body of arbitrary shape is held in equilibrium under the action of a hydrostatic pressure $P(>0)$ per unit area of the surface of the deformed configuration as shown in Figure 6,

$$t(n)=-Pn \qquad (3.2.4)$$

As another example, consider a block extended in the 1-direction and maintained in equilibrium by applying a uniform tension T per unit area of the deformed configuration on its end faces as shown in Figure 7. Suppose that after the deformation, the block occupies the region

$$-a\le x_1\le a, -b\le x_2\le b, -c\le x_3\le c$$

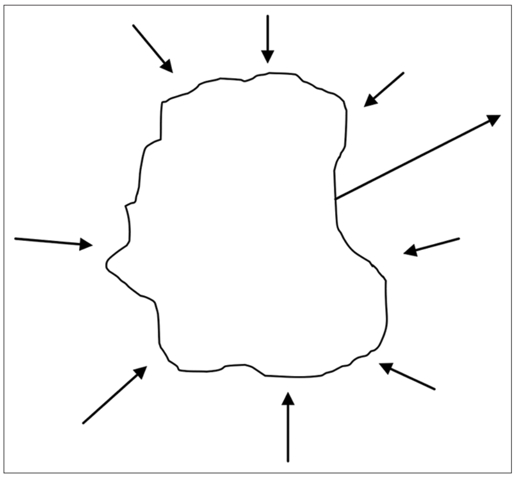

Figure 6: Problems with Boundary Conditions

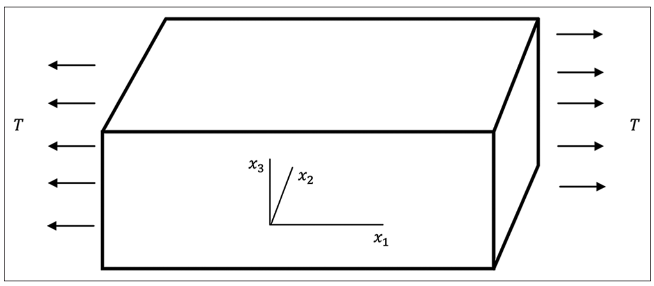

Figure 7: Basic equations with boundary conditions

The end faces $x_1=\pm a$ are both perpendicular to the 1-direction but they have different outward unit normal. On the face $x_1=-a$ the outward unit normal is $-e_1$ so that our boundary condition specifies $t(-e_1)$ and requires that

$$t(-e_1)=-Te_1 \tag{3.2.5}$$

However, at any point on this face, $t(-e_1)=-t(e_1)=-(T_{11},T_{12},T_{13})$, so that an equivalent statement is

$$T_{11}=T,\ T_{12}=T_{13}=0 \tag{3.2.6}$$

Likewise, on the face $x_1=a$,

$$t(e_1)=Te_1 \tag{3.2.7}$$

which again rise to (3.2.6). Since we are not applying any forces to the remaining faces, the applied surface traction on these faces is zero. Such boundaries are said to be traction free. We use this term rather than "stress free" since in many cases the surface is not "stress free" even though it is free of applied traction. The vanishing of the applied traction implies

$$t(e_2)=0 \text{ on } x_2=b,\ t(-e_2)=0 \text{ on } x_2=-b \tag{3.2.8}$$

and so the stresses T_{21}, T_{22}, and T_{23} are zero on $x_2=\pm b$, but in these faces $T_{11}\neq 0$. Similarly, the stresses T_{31}, T_{32}, and T_{33} are zero on $x_3=\pm c$ but again $T_{11}\neq 0$.

Consider next a hollow circular cylinder as shown in Figure 8, which is held in equilibrium under suitable surface tractions. In the deformed configuration, let the inner and outer radii be a_1

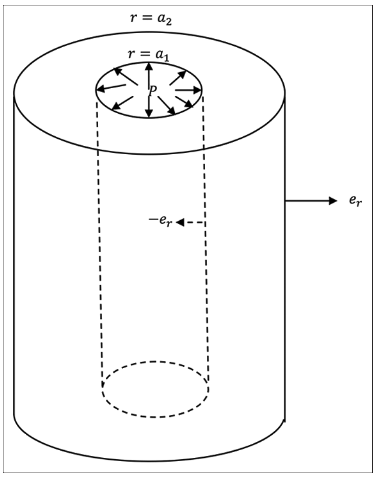

Figure 8: A hollow circular cylinder in equilibrium under suitable surface tractions

and a_2 respectively. The surface $r=a_1$ is subjected to a uniform pressure P and the outer surface $r=a_2$ is traction free. Traction is also applied to the end faces. On the surface $r=a_1$, the outward unit normal is $-e_r$ so that our boundary condition on this surface specifies $t(-e_r)$ and requires that

$$t(-e_r)=Pe_r \tag{3.2.9}$$

on $r=a_1$. The outer surface is traction free provided

$$t(e_r)=0 \tag{3.2.10}$$

on $r=a_2$. Now, $t_i=T_{ij}\,n_i$ and $e_r=r^{-1}\,(x_1,x_2,0)$, so that on $r=a_2$,

$$t_i\,(e_r)=T_i\alpha\, x\alpha/a_2,\ \alpha=1,2 \tag{3.2.11}$$

Thus, a statement equivalent to (3.2.10) is

$$t_1=T_{11}\frac{x_1}{a_2}+T_{12}\frac{x_2}{a_2}=0,\ t_2=T_{21}\frac{x_1}{a_2}+T_{22}\frac{x_2}{a_2}=0,$$
$$t_3=T_{31}\frac{x_1}{a_2}+T_{32}\frac{x_2}{a_2}=0 \tag{3.2.12}$$

The relation (3.2.9) may be expanded similarly.

We see from the above examples that in a particular problem, we are required to solve (3.2.1) and (3.2.2) subject to prescribed boundary conditions. In writing down these conditions, it is important to be able to identify the outward unit normal to the surface under consideration and also to realize which components of the stress tensor are being specified by the applied surface tractions. In the above examples, the body is considered in the deformed configuration and T and P are measured per unit area in this state. It is, therefore, appropriate to use the Cauchy stress tensor T. In some situations, the applied forces may be measured per unit area of the reference configuration, in which case the PiolaKirchoff stress tensors are more useful.

For an incompressible material, there is no volume change, so that we have the condition

$$J=1 \tag{3.2.13}$$

and the density has the value ρ_0 for all time. Equation (3.2.1), therefore, reduces to an identity and the equations which have to be satisfied in this case are (3.2.13) and (3.2.2),

$$T_{ij} = -p\delta_{ij} + 2W_1 B_{ij} - 2W_2 B_{ij}^{-1} \qquad (3.2.14)$$

p being an unknown scalar. As we shall see later, in a particular problem is determined by equilibrium equations (3.2.2) and the specified boundary conditions.

CONCLUSION

This research paper has extensively discussed the mathematical theory of Extension and Contraction in the theory of elasticity and in the end tries to make this classical mathematical theory more meaningful in the process of solving some difficult elasticity problems in science and Engineering.

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