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# Common Fixed Point Theorems for Non-linear Contractive Mapping in Hilbert Space

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## ABSTRACT

In this manuscript, we prove the existence and uniqueness of a common fixed point for a continuous self-mapping over a closed subset of Hilbert space satisfying non-linear rational type contraction condition.

**Key words:** Hilbert space, Fixed point, Closed subset, Completeness

## INTRODUCTION

Fixed point theory is an important tool used in solving a variety of problems in control theory, economic theory, global analysis, and non-linear analysis. Several authors worked on the applications, generalizations, and extensions of the Banach contraction principle[1] in different directions by either weakening the hypothesis, using different setups or considering different mappings.

Banach contraction principle has been extended and generalized by several authors such as Kannan[2] who investigated the extension of Banach fixed point theorem by removing the completeness of the space with different conditions. Chatterji[3] considered various contractive conditions for self-mappings in metric space. Dass and Gupta[5] also investigated the rational type of contractions to obtain a unique fixed point in complete metric space. Fisher[10] developed the approach of Khann[2] and proved analogous results involving two mappings on a complete metric space.

## PRELIMINARIES

The main aim of this paper is to prove the existence and uniqueness of a common fixed point for a continuous self-mapping $T$, some positive integers $r,s$ of a pair of continuous self-mapping $T^r, T^s$. These results generalized and extend the results of[2-13] and[10] here are the lists of some of the results that motivated our results.

**Theorem 2.1.**[2] proved that "If $f$ is self-mapping of a complete metric space $X$ into itself satisfying $d(x,y)\leq\alpha[d(Tx,x)+d(Ty,y)]$, for all $x,y\in X$ and $\alpha\in[0,1/2)$, then $f$ has a unique fixed point in $X$".

**Theorem 2.2.**[10] Prove the result with $d(x,y)\leq\alpha[d(Tx,x)+d(Ty,y)]+\beta d(x,y)$, for all $x,y \in X$ and $\alpha,\beta \in[0,1/2)$, then $f$ has a unique fixed point in $X$".

**Theorem 2.3.**[8] Proved the common fixed point theorems in Hilbert space with the following condition.

$$\|Tx-Ty\|\leq\alpha\left\{\frac{\|x-Tx\|^2+\|y-Ty\|^2}{\|x-Tx\|+\|y-Ty\|}\right\}+\beta\|x-y\| \text{ for}$$

all $x,y \in S, x\neq y$, $\alpha \in[0,1/2)$, $\beta\geq 0$ and $2\alpha+\beta<1$.

**Theorem 2.4.**[9] Proved the fixed point theorems in Hilbert space with the following condition.

$$\|Tx-Ty\|\leq\alpha\left\{\frac{\|x-Tx\|^2+\|y-Ty\|^2}{\|x-Tx\|+\|y-Ty\|}\right\}+$$

$$\beta\left\{\frac{\|x-Ty\|^2+\|y-Tx\|^2}{\|x-Ty\|+\|y-Tx\|}\right\}+\gamma\|x-y\|$$

for all $x,y \in S, x\neq y$, $0\leq \alpha,\beta < 1/2$, $0\leq\gamma$ $2\alpha$ and $2\beta+\gamma<1$.

**Theorem 2.5.**[14] Proved the unique fixed point theorems in Hilbert space with the following condition.

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$$\|Tx-Ty\| \le a_1 \left\{\frac{\|x-Tx\|^2+\|y-Ty\|^2+\|x-Ty\|^2+\|y-Tx\|^2}{\|x-Tx\|+\|y-Ty\|+\|x-Ty\|+\|y-Tx\|}\right\}+a_2 \left\{\frac{\|x-Tx\|^2+\|y-Ty\|^2}{\|x-Tx\|+\|y-Ty\|}\right\}+a_3\left\{\frac{\|x-Ty\|^2+\|y-Tx\|^2}{\|x-Ty\|+\|y-Tx\|}\right\}+a_4\|x-y\|$$

For all $x,y \in S$ and $4\alpha_1,\alpha_2,\alpha_3,\alpha_4 \ge 0$and $4\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 < 1$.

**Theorem 2.6.**[7] Proved the unique common fixed point theorems for two self-mappings $T_1, T_2$ of closed subset $X$ in Hilbert space satisfying the inequality.

$$\|T_1x-T_2y\| \le a_1 \frac{x-T_1x\left[1+\|y-T_2y\|\right]}{1+\|x-y\|}+a_2 \frac{\|y-T_2y\|\left[1+\|y-T_1x\|\right]}{1+\|x-y\|}+ a_3\frac{\|x-T_2y\|\left[1+\|y-T_1x\|\right]}{1+\|x-y\|}+ a_4\frac{\|x-y\|\left[1+\|T_1x-T_2y\|\right]}{1+\|x-y\|} +a_5\frac{\|x-y\|\left[1+\|x-T_1x\|\right]}{1+\|y-T_2y\|}+ a_6\frac{\|x-T_1x\|\left[1+\|x-T_2y\|\right]}{1+\|y-T_2y\|}+ a_7\frac{\|x-T_2y\|\left[1+\|T_1x-T_2y\|\right]}{1+\|x-y\|}+a_8 \frac{\|x-y\|\left[1+\|x-T_2y\|\right]}{1+\|x-y\|}+ a_9\frac{\|x-T_1x\|+\|y-T_2y\|+\|x-y\|}{1+\|x-T_1x\|\|x-T_2y\|\|y-T_2y\|\|x-y\|}+ a_{10}\frac{\|x-T_2y\|^2+\|y-T_1x\|^2}{\|x-T_2y\|+\|y-T_1x\|}+a_{11}\left[\|x-T_1x\|+\|y-T_2y\|\right] +a_{12}\left[\|x-T_2y\|+\|y-T_1x\|\right]+a_{13}\|x-y\|.$$

For all $x,y \in X$ and$x \neq y$, where $a_i$ ($i$=1,2,3,…,13) are non-negative reals with $0 \le \sum_{i=1}^{8} a_i + 3a_9 + 2\sum_{i=10}^{12} a_i + a_{13} < 1$.

**Theorem 2.7.**[6] Proved the unique common fixed point theorems for two self-mappings $T_1$ $T_2$ of closed subset X in Hilbert space satisfying the inequality.

$$\|T_1x-T_2y\|^2 \le \alpha\frac{\|x-T_2y\|^2\left[1+\|y-T_1x\|^2\right]}{1+\|x-y\|^2}+ \beta\frac{x-y^2\left[1+\|x-T_2y\|^2\right]}{1+\|x-y\|^2}+\gamma\left[\|y-T_1x\|^2+\|x-T_2y\|^2\right] +\delta\|x-y\|^2$$

for all $x, y \in S, x \neq y$ where $\alpha,\beta,\gamma,\delta$ and $4\alpha + \beta + 4\gamma + \delta < 1$.

**Theorem 2.8.**[6] Proved the unique common fixed point theorems for two self-mappings $T_1, T_2$ and $p,q$ (positive integers) of closed subset X in Hilbert space satisfying the inequality.

$$\|T_1^p x-T_2^q y\|^2 \le \alpha\frac{\|x-T_2^q y\|^2\left[1+\|y-T_1^p x\|^2\right]}{1+\|x-y\|^2}+ \beta\frac{\|x-y\|^2\left[1+\|x-T_2^q y\|^2\right]}{1+\|x-y\|^2}+\gamma\left[\|y-T_1^p x\|^2+\|x-T_2^q y\|^2\right] +\delta\|x-y\|^2$$

for all $x,y \in S, x\neq y$, where $\alpha,\beta,\gamma,\delta$ and $4\alpha + \beta + 4\gamma + \delta < 1$.

## MAIN RESULTS

**Theorem 3.1.** Let $X$ be a closed subset of a Hilbert space and $T$: $X \to X$ be a continuous self-mapping satisfying the following inequality.

$$\|Tx-Ty\| \le c_1\frac{\|x-Tx\|\left[1+\|y-Ty\|\right]}{1+\|x-y\|} +c_2\frac{\|y-Ty\|\left[1+\|y-Tx\|\right]}{1+\|x-y\|}+ c_3\frac{\|x-Ty\|\left[1+\|y-Tx\|\right]}{1+\|x-y\|}+c_4\frac{x-y\left[1+\|Tx-Ty\|\right]}{1+\|x-y\|}$$

$$+c_5 \frac{\|x-y\|\left[1+\|x-Tx\|\right]}{1+\|y-Ty\|}+c_6 \frac{\|x-Tx\|\left[1+\|x-Ty\|\right]}{1+\|y-Ty\|}+$$

$$c_7 \frac{\|x-Ty\|\left[1+\|Tx-Ty\|\right]}{1+\|x-y\|}+c_8 \frac{\|x-y\|\left[1+\|x-Ty\|\right]}{1+\|x-y\|}$$

$$+c_9 \frac{\|x-Tx\|+\|y-Ty\|+\|x-y\|}{1+\|x-Tx\|\|x-Ty\|\|y-Ty\|\|x-y\|}$$

$$+c_{10} \frac{\|x-Ty\|^2+\|y-Tx\|^2}{\|x-Ty\|+\|y-Tx\|}+c_{11}\left[\|x-Tx\|+\|y-Ty\|\right]$$

$$+c_{12}\left[\|x-Ty\|+\|y-Tx\|\right]+c_{13}\|x-y\|$$

for all $x,y \in X$ and $x \neq y$, where $c_i$ ($i$=1,2,3,…,13) are non-negative real numbers with $0 \leq \sum_{i=1}^{8} c_i + 3c_9 + 2\sum_{i=10}^{12} c_i + c_{13} < 1$. Then, $T$ has a unique fixed point in $X$.

**Proof.** We construct a sequence $\{x_n\}$ for an arbitrary point $x_0 \in X$ defined as follows
$x_{2n+1} = Tx_{2n}$, $x_{2n+2}=Tx_{2n}+1$, for $n$=1,2,3.,
We show that the sequence $\{x_n\}$ is a Cauchy sequence in $X$

$$\|x_{2n+1}\|-\|x_{2n}\|=\|Tx_{2n}\|-\|Tx_{2n-1}\|$$

$$\leq c_1 \frac{\|x_{2n}-Tx_{2n}\|\left[1+\|x_{2n-1}\|-\|Tx_{2n-1}\|\right]}{1+\|x_{2n}\|-\|x_{2n-1}\|}$$

$$+c_2 \frac{\|x_{2n-1}\|-\|Tx_{2n-1}\|\left[1+\|x_{2n-1}\|-\|Tx_{2n}\|\right]}{1+\|x_{2n}\|-\|x_{2n-1}\|}$$

$$+c_3 \frac{\|x_{2n}-Tx_{2n-1}\|\left[1+\|x_{2n-1}\|-\|Tx_{2n}\|\right]}{1+\|x_{2n}\|-\|x_{2n-1}\|}$$

$$+c_4 \frac{\|x_{2n}-x_{2n-1}\|\left[1+\|Tx_{2n}\|-\|Tx_{2n-1}\|\right]}{1+\|x_{2n}\|-\|x_{2n-1}\|}$$

$$+c_5 \frac{\|x_{2n}-x_{2n-1}\|\left[1+\|x_{2n}\|-\|Tx_{2n}\|\right]}{1+\|x_{2n-1}\|-\|Tx_{2n-1}\|}+$$

$$c_6 \frac{\|x_{2n}-Tx_{2n}\|\left[1+\|x_{2n}\|-\|Tx_{2n-1}\|\right]}{1+\|x_{2n-1}\|-\|Tx_{2n-1}\|}$$

$$+c_7 \frac{\|x_{2n}-Tx_{2n-1}\|\left[1+\|Tx_{2n}\|-\|Tx_{2n-1}\|\right]}{1+\|x_{2n}\|-\|x_{2n-1}\|}$$

$$+c_8 \frac{\|x_{2n}-x_{2n-1}\|\left[1+\|x_{2n}\|-\|Tx_{2n-1}\|\right]}{1+\|x_{2n}\|-\|x_{2n-1}\|}$$

$$+c_9 \frac{\|x_{2n}-Tx_{2n}\|+\|x_{2n-1}-Tx_{2n-1}\| +\|x_{2n}-x_{2n-1}\|}{1+\|x_{2n}-Tx_{2n}\|\|x_{2n}-Tx_{2n-1}\| \|x_{2n-1}-Tx_{2n-1}\|\|x_{2n}-x_{2n-1}\|}$$

$$+c_{10} \frac{\|x_{2n}-Tx_{2n-1}\|^2+\|x_{2n-1}-Tx_{2n}\|^2}{\|x_{2n}-Tx_{2n-1}\|+\|x_{2n-1}-Tx_{2n}\|}$$

$$+c_{11}\left[\|x_{2n}-Tx_{2n}\|+\|x_{2n-1}-Tx_{2n-1}\|\right]$$

$$+c_{12}\left[\|x_{2n}-Tx_{2n-1}\|+\|x_{2n-1}-Tx_{2n}\|\right]+c_{13}\|x_{2n}-x_{2n-1}\|$$

This implies

$$\|x_{2n+1}-x_{2n}\|=a(n)\|Tx_{2n}\|-\|Tx_{2n-1}\|,$$

where

$$a(n)=\frac{B_1+\left(c_2+2c_9+c_{10}+c_{11}+c_{12}\right) \|x_{2n}-x_{2n-1}\|}{B_2+\left(1-c_1-c_2-c_4-c_5-c_9-c_{10}-c_{11}-c_{12}\right) \|x_{2n}-x_{2n-1}\|},$$

$B_1 = c_2+c_4+c_5+c_8+2c_9+c_{10}+c_{11}+c_{12}+c_{13}$ and $B_2$=1-$c_1$-$c_6$-$c_9$-$c_{10}$-$c_{11}$-$c_{12\ S}$
Clearly, δ=a (n)<1,∀n=1,2,3,.., we get

$$\|x_{n+1}-x_n\|=\delta\|x_n-x_{n-1}\|,$$

Recursively, we have

$$\|x_{n+1}-x_n\|=\delta^n\|x_1-x_0\|, n \geq 1,$$

Takingn→∞, we have $\|x_{n+1}-x_n\| \to 0$.

Hence, $\{x_n\}$ is a Cauchy sequence in $X$ and so it has a limit λ in $X$. Since the sequences $\{x_{2n+1}\}=\{T_{x2n}\}$ and $\{x_{2n+2}\}=\{Tx_{2n+1}\}$ are subsequences of $\{x_n\}$, and also these subsequences have the same limit λ in $X$. We now show that λ is a common fixed point of $T$
Now consider the following inequality

$$\|\lambda - T\lambda\| = \|\lambda - x_{2n+2}\| + \|x_{2n+2} - T\lambda\| = (\lambda - x_{2n+2}) + (x_{2n+2} - T\lambda) \le \|\lambda - x_{2n+2}\| + \|T\lambda - Tx_{2n+1}\|$$

$$\le c_1 \frac{\|\lambda - T\lambda\|\left[\|1 + x_{2n+1} - Tx_{2n+1}\|\right]}{1 + \|\lambda - x_{2n+1}\|} + c_2 \frac{\|x_{2n+1} - Tx_{2n+1}\|\left[\|1 + x_{2n+1} - T\lambda\|\right]}{1 + \|\lambda - x_{2n+1}\|}$$

$$c_3 \frac{\|\lambda - Tx_{2n+1}\|\left[\|1 + x_{2n+1} - T\lambda\|\right]}{1 + \|\lambda - x_{2n+1}\|} + c_4 \frac{\|\lambda - x_{2n+1}\|\left[\|1 + T\lambda - Tx_{2n+1}\|\right]}{1 + \|\lambda - x_{2n+1}\|}$$

$$+ c_5 \frac{\|\lambda - x_{2n+1}\|\left[1 + \|\lambda - T\lambda\|\right]}{1 + \|x_{2n+1} - Tx_{2n+1}\|} + c_6 \frac{\|\lambda - T\lambda\|\left[1 + \|\lambda - Tx_{2n+1}\|\right]}{1 + \|x_{2n+1} - Tx_{2n+1}\|}$$

$$+ c_7 \frac{\|\lambda - Tx_{2n+1}\|\left[1 + \|T\lambda - Tx_{2n+1}\|\right]}{1 + \|\lambda - x_{2n+1}\|} + c_8 \frac{\|\lambda - x_{2n+1}\|\left[1 + \|\lambda - Tx_{2n+1}\|\right]}{1 + \|\lambda - x_{2n+1}\|}$$

$$+ c_9 \frac{\|\lambda - T\lambda\| + \|x_{2n+1} - Tx_{2n+1}\| + \|\lambda - x_{2n+1}\|}{1 + \|\lambda - T\lambda\|\|\lambda - Tx_{2n+1}\|\|x_{2n+1} - Tx_{2n+1}\|\|\lambda - x_{2n+1}\|} + c_{10} \frac{\|\lambda - Tx_{2n+1}\|^2 + \|x_{2n+1} - T\lambda\|^2}{\|\lambda - Tx_{2n+1}\| + \|x_{2n+1} - T\lambda\|}$$

$$+ c_{11}\left[\|\lambda - T\lambda\| + \|x_{2n+1} - Tx_{2n+1}\|\right] + c_{12}\left[\|\lambda - Tx_{2n+1}\| + \|x_{2n+1} - T\lambda\|\right] + c_{13}\|\lambda - x_{2n+1}\|.$$

Letting n→∞, we get ||λ-Tλ|| ≤ $(c_1+c_6+c_9+c_{10}+c_{11}+c_{12})$||λ-Tλ||, since $c_1+c_6+c_9+c_{10}+c_{11}+c_{12}<1$, hence λ = Tλ.
Similarly, from the hypothesis, we get λ = Tλ by considering the following

$$\|\lambda - T\lambda\| = \|(\lambda - x_{2n+1}) + (x_{2n+1} - T\lambda)\|.$$

We now show that λ is a unique fixed point of *T*. Suppose that t (λ≠t) is also a common fixed point of T.
Then, by the hypothesis, we get

$$\|\lambda - t\| = \|T\lambda - Tt\| \le c_1 \frac{\|\lambda - T\lambda\|\left[1 + \|t - Tt\|\right]}{1 + \|\lambda - t\|} + c_2 \frac{\|t - Tt\|\left[1 + \|t - T\lambda\|\right]}{1 + \|\lambda - t\|} + c_3$$

$$\frac{\|\lambda - Tt\|\left[1 + \|t - T\lambda\|\right]}{1 + \|\lambda - t\|} + c_4 \frac{\|\lambda - t\|\left[1 + \|T\lambda - Tt\|\right]}{1 + \|\lambda - t\|} + c_5 \frac{\lambda - t\left[1 + \|\lambda - T\lambda\|\right]}{1 + \|t - Tt\|}$$

$$+ c_6 \frac{\|\lambda - T\lambda\|\left[1 + \|\lambda - Tt\|\right]}{1 + \|t - Tt\|} + c_7 \frac{\|\lambda - Tt\|\left[1 + \|T\lambda - Tt\|\right]}{1 + \|\lambda - t\|} + c_8 \frac{\|\lambda - t\|\left[1 + \|\lambda - Tt\|\right]}{1 + \|\lambda - t\|}$$

$$+ c_9 \frac{\|\lambda - T\lambda\| + \|t - Tt\| + \|\lambda - t\|}{1 + \|\lambda - T\lambda\|\|\lambda - Tt\|\|t - Tt\|\|\lambda - t\|} + c_{10} \frac{\|\lambda - Tt\|^2 + \|t - T\lambda\|^2}{\|\lambda - Tt\| + \|t - T\lambda\|}$$

$$+ c_{11}\left[\|\lambda - T\lambda\| + \|t - Tt\|\right] + c_{12}\left[\|\lambda - Tt\| + \|t - T\lambda\|\right] + c_{13}\|\lambda - t\|.$$

Thus,

$$\|\lambda - t\| \le (c_3 + c_4 + c_5 + c_7 + c_8 + c_9 + c_{10} + 2c_{12} + c_{13})$$
$$\|\lambda - t\| < \|\lambda - t\|$$

a contradiction. Hence λ=*t* (common fixed point λ is unique in *X*).

**Theorem 3.2.** Let *X* be a closed subset of a Hilbert space and *T: X*→*X* be a continuous self mapping satisfying

$$\|T^r x - T^s y\| \le c_1 \frac{\|x - T^r x\|\left[1 + \|y - T^s y\|\right]}{1 + \|x - y\|} + c_2 \frac{\|y - T^s y\|\left[1 + \|y - T^r x\|\right]}{1 + \|x - y\|} +$$

$$c_3 \frac{\|x - T^s y\|\left[1 + \|y - T^r x\|\right]}{1 + \|x - y\|} + c_4 \frac{\|x - y\|\left[1 + \|T^r x - T^s y\|\right]}{1 + \|x - y\|}$$

$$+c_5\frac{\|x-y\|\left[1+\|x-T^r x\|\right]}{1+\|y-T^s y\|}+c_6\frac{\|x-T^r x\|\left[1+\|x-T^s y\|\right]}{1+\|y-T^s y\|}$$

$$+c_7\frac{\|x-T^s y\|\left[1+\|T^r x-T^s y\|\right]}{1+\|x-y\|}+c_8\frac{\|x-y\|\left[1+\|x-T^s y\|\right]}{1+\|x-y\|}$$

$$+c_9\frac{\|x-T^r x\|+\|y-T^s y\|+\|x-y\|}{1+\|x-T^r x\|\|x-T^s y\|\|y-T^s y\|\|x-y\|}$$

$$+c_{10}\frac{\|x-T^s y\|^2+\|y-T^r x\|^2}{\|x-T^s y\|+\|y-T^r x\|}$$

$$+c_{11}\left[\|x-T^r x\|+\|y-T^s y\|\right]$$

$$+c_{12}\left[\|x-T^s y\|+\|y-T^r x\|\right]+c_{13}\|x-y\|$$

for all x, y∈ $X$ and x≠y, where $c_i$ ($i$=1,2,3,…,13) are non-negative real numbers with

$0\le\sum_{i=1}^{8}c_i+3c_9+2\sum_{i=10}^{12}c_i+c_{13}<1$ and $r,s$ are two positive integers. Then $T$ has a unique fixed point in $X$.

**Proof.** From theorem 3.1.$T^r$ and $T^s$ have a unique common fixed point λ∈X, so that $T^r$ λ=λ and $T^s$ λ=λ.

From $T^r$ (Tλ)=$T$($T^r$ λ)=Tλ, it follows that Tλ is a fixed point of $T^r$. But λ is a unique fixed point of $T^r$. Therefore Tλ=λ.

Similarly, we can get $T$λ=λ from $T^s$(Tλ)=T($T^s$λ)=Tλ. Hence λ is a unique fixed point of $T$.

Now, we show uniqueness. Let $t$ be another fixed point of $T$, so that $Tt$=$t$. then from the hypothesis, we have

$$\|\lambda-t\|=\|T^r\lambda-T^s t\|\le c_1\frac{\|\lambda-T^r\lambda\|\left[1+\|t-T^r t\|\right]}{1+\|\lambda-t\|}$$

$$+c_2\frac{\|t-T^s t\|\left[1+\|t-T^r\lambda\|\right]}{1+\|\lambda-t\|}+$$

$$c_3\frac{\|\lambda-T^s t\|\left[1+\|t-T^r\lambda\|\right]}{1+\|\lambda-t\|}+c_4\frac{\|\lambda-t\|\left[1+\|T^r\lambda-T^s t\|\right]}{1+\|\lambda-t\|}$$

$$+c_5\frac{\|\lambda-t\|\left[1+\|\lambda-T^r\lambda\|\right]}{1+\|t-T^s t\|}$$

$$+c_6\frac{\|\lambda-T^r\lambda\|\left[1+\|\lambda-T^s t\|\right]}{1+\|t-T^s t\|}$$

$$+c_7\frac{\|\lambda-T^s t\|\left[1+\|T^r\lambda-T^s t\|\right]}{1+\|\lambda-t\|}+c_8\frac{\|\lambda-t\|\left[1+\|\lambda-T^s t\|\right]}{1+\|\lambda-t\|}$$

$$+c_9\frac{\|\lambda-T^r\lambda\|+\|t-T^s t\|+\|\lambda-t\|}{1+\|\lambda-T^r\lambda\|\|\lambda-T^s t\|\|t-T^s t\lambda\|-t}$$

$$+c_{10}\frac{\|\lambda-T^s t\|^2+\|t-T^r\lambda\|^2}{\|\lambda-T^s t\|+\|t-T^r\lambda\|}+c_{11}\left[\|\lambda-T^r\lambda\|+\|t-T^s t\|\right]$$

$$+c_{12}\left[\|\lambda-T^s t\|+\|t-T^r\lambda\|\right]+c_{13}\|\lambda-t\|.$$

Therefore, we have

$$\|\lambda-t\|\le\left(c_3+c_4+c_5+c_7+c_8+c_9+c_{10}+2c_{12}+c_{13}\right)$$

$$\|\lambda-t\|<\|\lambda-t\|$$

Implies λ=$t$, since $c_3+c_4+c_5+c_7+c_8+c_9+c_{10}+2c_{12}+c_{13}<1$.

Hence, λ is a unique common fixed point of Tin $X$.

Example 3.3. Let $T$: [0,1]→[0,1] be a mapping defined by T$x$=$x^3/6$, for all $x$∈[0,1]with usual norm ||$x$-y||=|$x$-y|, for all $x$∈[0,1].

**Proof.** From theorem 3.1, we have

$$\|Tx-Ty\|=\left\|\frac{x^3}{6}-\frac{y^3}{6}\right\|\le c_1\frac{\left\|x-\frac{x^3}{6}\right\|\left[1+\left\|y-\frac{y^3}{6}\right\|\right]}{1+\|x-y\|}$$

$$+c_2\frac{\left\|y-\frac{y^3}{6}\right\|\left[1+\left\|y-\frac{x^3}{6}\right\|\right]}{1+\|x-y\|}+$$

$$c_3\frac{\left\|x-\frac{y^3}{6}\right\|\left[1+\left\|y-\frac{x^3}{6}\right\|\right]}{1+\|x-y\|}+c_4\frac{\|x-y\|\left[1+\left\|\frac{x^3}{6}-\frac{y^3}{6}\right\|\right]}{1+\|x-y\|}$$

$$+c_5\frac{\|x-y\|\left[1+\left\|x-\frac{x^3}{6}\right\|\right]}{1+\left\|y-\frac{y^3}{6}\right\|}$$

$$+c_6 \frac{\left\|x-\frac{x^3}{6}\right\|\left[1+\left\|x-\frac{y^3}{6}\right\|\right]}{1+\left\|y-\frac{y^3}{6}\right\|}$$

$$+c_7 \frac{\left\|x-\frac{y^3}{6}\right\|\left[1+\left\|\frac{x^3}{6}-\frac{y^3}{6}\right\|\right]}{1+\|x-y\|}+$$

$$c_3 \frac{\left\|x-\frac{y^3}{6}\right\|\left[1+\left\|y-\frac{x^3}{6}\right\|\right]}{1+\|x-y\|}+c_4 \frac{\|x-y\|\left[1+\left\|\frac{x^3}{6}-\frac{y^3}{6}\right\|\right]}{1+\|x-y\|}$$

$$+c_5 \frac{\|x-y\|\left[1+\left\|x-\frac{x^3}{6}\right\|\right]}{1+\left\|y-\frac{y^3}{6}\right\|}$$

$$+c_6 \frac{\left\|x-\frac{x^3}{6}\right\|\left[1+\left\|x-\frac{y^3}{6}\right\|\right]}{1+\left\|y-\frac{y^3}{6}\right\|}$$

$$+c_7 \frac{\left\|x-\frac{y^3}{6}\right\|\left[1+\left\|\frac{x^3}{6}-\frac{y^3}{6}\right\|\right]}{1+\|x-y\|}+$$

$$c_8 \frac{\|x-y\|\left[1+\left\|x-\frac{y^3}{6}\right\|\right]}{1+\|x-y\|}$$

$$+c_9 \frac{\left\|x-\frac{x^3}{6}\right\|+\left\|y-\frac{y^3}{6}\right\|+\|x-y\|}{1+\left\|x-\frac{x^3}{6}\right\|\left\|x-\frac{y^3}{6}\right\|\left\|y-\frac{y^3}{6}\right\|\|x-y\|}$$

$$+c_{10} \frac{\left\|x-\frac{y^3}{6}\right\|^2+\left\|y-\frac{x^3}{6}\right\|^2}{\left\|x-\frac{y^3}{6}\right\|+\left\|y-\frac{x^3}{6}\right\|}+c_{11}\left[\left\|x-\frac{x^3}{6}\right\|+\left\|y-\frac{y^3}{6}\right\|\right]$$

$$+c_{12}\left[\left\|x-\frac{y^3}{6}\right\|+\left\|y-\frac{x^3}{6}\right\|\right]+c_{13}\|x-y\|$$

Continue with the procedure in the proof of theorem 3.1, we get that 0 is the only fixed point of $T$.

## CONCLUSION

In this paper, we proved the existence and uniqueness of a common fixed point for a continuous self mapping, $T$ some positive integers $r,s$ of a pair of continuous self mapping $T^r, T^s$ of Hilbert space. These results generalized and extend the results of some literatures.


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