ISSN 2581-3463

AJMS
Asian Journal of Mathematical Sciences

RESEARCH ARTICLE

On Extension Theorems in Ordinary Differential Systems of Equations

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Received: 15-05-2022; Revised: 10-06-2022; Accepted: 12-07-2022

ABSTRACT

Given the differential system $x' = f(t, x)$; $f: J \times M \to R^n$, we establish the Peano's theorem on existence of the solution plus Picard Lindelof theorem on uniqueness of the solution. Using the two, we then worked on the extendibility of the solutions whose local existence is ensured by the above in a domain of open connected set producing the following results; If D is a domain of $R \times R_n$ and $F: D \to R_n$ is continuous and suppose that (t_0, x_0) is a point D and if the system has a solution $x(t)$ defined on a finite interval (a, b) with $t \in (a, b)$ and $x(t_0) = x_0$, then whenever f is bounded and D, the limits

$$x\left(a^+\right) = \lim_{t\to a^+} x(t)$$

$$x\left(b^-\right) = \lim_{t\to b^-} x(t)$$

Exists as finite vectors and if the point $(a, x(a^+)),(b, x(b^-))$ is in D then $x(t)$ is extendable to the point $t = a$ $(t = b)$. In addition to this are other results as could be seen in major sections of this work.

Key words: Differential systems, Continuous maps, Existence and uniqueness of solutions, Extension of solutions

INTRODUCTION

Presented in this section is a system of the form

$$x' = f(t, x) \qquad (1.1)$$

Where $f: J \times M \to R^n$ is continuous

Here, J is an interval of R and M is a subset of R^n. Furthermore, presented here are the basic theorems on existence and uniqueness conditions for solution of any given differential system particularly those of the type in (1)

Theorem 1.1 (The Peano's Theorem on Existence of Solutions)

Let (t_0, x_0) be a given point in $R \times R^n$. Let $J = \{t_0 - a, t_0 + a\}$, $D = \{x \in R^n: |x - x_0| \leq b\}$, where a, b are two positive constants in (1.1). Assume the following: $F: J \times D \to R^n$ is continuous with $|F(t, u)| \leq L, (t, u) \in J \times D$, where L is a positive constant. Then, there exists a solution $x(t)$ of (1.1) with the following property: $x(t)$ is defined and satisfied (1.1) on $S = \{t \in J: |t - t_0| \leq \alpha\}$ with $\alpha = \min\{\alpha, b/L\}$. Moreover $x(t_0) = x_0$ and $|x(t) - x_0| \leq b$ for all $t \in \{t_0 - \alpha, t_0 + \alpha\}$. For proof see [1].

Theorem 1.2 (Picard's Lindelof Theorem on Uniqueness]

Consider system (1.1) under the assumptions of theorem (1.1). Let $F: J \times D \to R^n$ satisfy
$|F(t, x_1) - F(t, x_0)| \leq K|x_1 - x_0|$
For every $(t, x_1), (t, x_2) \in J \times D$, where K is a positive constant, then there exists a unique solution $x(t)$ satisfying the conclusion of theorem (1.1). The proof is in [1]

Theorem 1.3[1-3]

Let $F: (a \times b) \times D \to R^n$ be continuous, where $D = \{u \in R^n; |u - x_0| \leq r\}$, x_0 a fixed point R^n and r, a positive constant.

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Let $\{|F(t, u) - F(t, u_0| \leq I(t)|u_1 - u_0|\}$
$|F(t, u)| \leq m(t)$ For every $t \in (a, b)$, $u, u_0, u_1 \in D$, Where $I, m: [a, b] \to R_+$ are continuous and integrable in the improper sense on (a, b) choose the number $b \ni a_1 < b_1 < b$ so that

$$L = \int_{a^+}^{b_1} I(t)\,dt < 1, \int_{a^+}^{b_1} m(t)\,dt \leq r$$

Then, the integral equation has unique solution $x(t)$ on the interval $[a, b]$. This solution satisfies $x(a^+) = x_0$ and the system (1.1) on the interval $[a, b]$

EXTENSION

This section presents the study of extendibility or continuation of the solutions whose local existence is ensured by theorem (1.1) and (1.2). In what follows, a domain is an open connected set and we have the following definition.

Definition 2.1[4-6]

A solution $x(t)$, $t \in [a, b]$, $a < b < +\infty$ of system (1.1) is said to be extendable (continuable) to $t = b$ if there exists another solution $\bar{x}(t)$ $t \in [a, c]$, $c \geq b$, of the system (1.1) such that $\bar{x}(t) = x(t)$, $t \in [a, b]$.

A solution $x(t)$, $t \in [a, b]$, $(a < b < +\infty)$ of the system (1.1) is said to be extendable (continuable) to $t = c$, $a < b < +\infty$ if it is extendable (continuable) to $t = b$ and whenever we assume that $x(t)$ is a solution on $[a, d]$ for any $d \in [b, c]$ we can show that $x(t)$ is extendable (continuable) to $t = d$. Such a solution is extendable (continuable) $t = c$ forany $c > b$. Extension to the left can be defined in a similar manner.

Theorem 2.1[7-9]

Suppose that D is a domain of $R \times R^n$ and $F: D \to R^n$ is continuous. Let (t_0, x_0) be a point in D and assume that the system (1.1) has a solution $x(t)$ defined on a finite interval (a, b) with $t \in (a, b)$ and $x(t_0) = x_0$. Then, if F is bounded on D, the limits

$$\left.\begin{aligned} x(a^+) &= \lim_{t \to a^+} x(t) \\ x(b^-) &= \lim_{t \to b^-} x(t) \end{aligned}\right\} \tag{2.1}$$

Exists as finite vectors. If the point $(a, x(a^+))$, $(b, x(b^-))$ is in D, then $x(t)$ is extendable to the point $t = a$ $(t = b)$.

Proof: To show that the first limit (2.1) exists, we first note that

$$x(t) = x_0 + \int_{x_0}^{x} F(s, x(s))\,ds, t \in (a, b) \qquad *$$

Now, let $|F(t, x)| < L$ for $(t, x) \in D$ D
Where L is a positive constant
Then if $t_1, t_2 \in [c, b]$ we obtain

$$|x(t_1) - x(t_2)| \leq \int_1^{t_2} |F(s, x(s))|\,ds \leq L|t_1 - t_2|$$

Thus, $x(t_1) - x(t_2)$ converges to zero as t_1, t_2 converges to the point $t = a$ from the right.
Applying the continuity condition for functions, we obtain our assertion.
Similarly, the second limit follow same argument let us assume now that the point $(b, x(b^-))$ belongs to D and consider the function

$$\begin{cases} x(t), t \in (a, b) \\ x(b^-), t = b \end{cases} \bar{x}(t) =$$

This function is a solution of (1.1) on $(a, b]$ In fact (*) implies

$$\bar{x}(t) = x_0 \int_o^c F(s, x(s))\,ds, t \in [a, b]$$

This in turn implies the existence of the left hand derivative $\bar{x}'(b)$ of $\bar{x}(t)$ as $t = b$

Thus, we have

$$\bar{x}'(b) = F(b, \bar{x}(b))$$

Which completes the proof for $t = b$. A similar argument holds for $t = a$

Remark 2.1

It should be noted that the point $(a, x(a'))$ is not in D, but $F(c, x(a'))$ can be defined so that F is continuous at $(a, x(a'))$ then $x(t)$ is extendable to $(a, x(a'))$. A similar situation exists at $(b, x(b^-))$.
The following extension theorem is needed for the proof of theorem 2.3

Theorem 2.2

Let $F: [a, b] \times R^n \to R^n$ be continuous and such that $\|F(t, x)\| \leq L$ for all $(t, x) \in [a, b] \times R$

Where L is a positive constant. Then every solution $x(t)$ of (1.1) is extendable to the point $t = b$.

Proof: Let $x(t)$ be a solution of (1.1) passing through the point $(t_0, x_0) \in [a, b] \times R^n$. Assume that $x(t)$ is defined on the interval (t_0, c) where c is same point with $c \leq b$. Then, as in the proof of theorem 2.1 above, $x(c^-)$ exists and $x(t)$ is extendable to the point $t = c$. If $c = b$, the proof is complete. If $c < b$, then Peano's theorem applied on $[c, b] \times D$ with D a sufficiently large closed ball with center at $x(c^-)$ ensures the existence of a solution such that $\bar{x}(t), t \in [c, b]$ such that $\bar{x}(c) = x(c^-)$

Thus, the function

$$\begin{cases} x(t), t \in [t_0, c] \\ \bar{x}(t),\ t \in [c, b] \end{cases} x(t) =$$

Is the required extension of $x(t)$.

Remark 2.2 [8-10]

The next theorem assures that the boundedness of every solution through a certain point in a certain sense and hence implies extendibility.

Theorem 2.3

Let F: $[a, b] \times M \rightarrow R^n$ be continuous, where M is closed ball $S = \{u \in R: |u| \leq r\}, r > 0$ (or R'). Assume that $(t_0, x_0) \in [a, b] \times M$ is given and that every solution $x(t)$ of (1.1) passing through (t_0, x_0) satisfies $|x(t)| < \lambda$ as long as it exists to the right of t_0. Here $0 < \lambda < r$ (or $0 < \lambda < \infty$). Then every solution $x(t)$ of (1.1) with $x(t_0) = x_0$ is extendable to the point $t = b$.

Proof: We give the proof for $M = S$.

If $M=R'$ the same proof holds and it is even easier. Let $x(t)$ be a solution of (1.1) with $x(t_0) = x_0$ and assume that $x(t)$ is defined on $[t_0, c]$ with $c < b$. Since F is continuous on $[a, b] \times S\lambda$ where

$$S = \{u \in \mathbb{R}^n : u \leq \lambda\}$$

There exists $L > 0$ such that

$\|F(t, x)\| \leq L$ for all $(t, x) \in [a, b] \times S$.

Now, consider the function

$$\begin{cases} F(t,x), (t,x) \in [a,b] \times S_\lambda \\ \dfrac{\lambda}{x} F\left(t, \dfrac{\lambda x}{x}\right), t \in [a,b] \end{cases} F(t,x) =$$

It is easy to see that F is continuous and such that $\|F(t, x)\| \leq L$ on $[a, b] \times R^n$. Consequently, theorem 2.2 above implies that every solution of the system

$$x' = F(t, x) \tag{2.1}$$

Is extendable to $t = b$.

Naturally, $x(t)$ is a solution of (2.1) defined on $[t, c]$ because $F\lambda = F$ for $\|x\| \leq \lambda$.

Therefore, there exists a solution $x(t)$, $t \in [t_0, b]$ of (2.1) such that $x(t) = (t)$, $t \in [t_0, c]$. Assume that there exists $t \in [c, b]$ such that $\|x(t)\| = \lambda$. Then, for some $t \in [c, t]$, $\|x(t)\| = \lambda$ and $\|x(t)\| < \lambda$ for all $t \in [t_0, t_2]$. Obviously $x(t)$ satisfies that system (2.1) on $[t_0, t_2]$.

This is a contradiction to our assumption. Thus $\|x(t)\| < \lambda$ for all $t \in [t_0, b]$ which implies that $x(t)$ is extendable to the point $t = b$.

MAIN RESULTS

Here, we discuss the existence of solutions on $R+$ by the approach of extension of solutions on R to $R+$. This we achieve more easily by the following theorems.

Theorem 3.1

Let $x(t)$, $t \in (t_0, t_1)$, $t_1 > t_0 \geq 0$ be an extendable to the right solution of the system (1.1). Then, there exists a non-extendable to the right solution of (1.1) which extends $x(t)$ to the right (which is $+\infty$).

Theorem 3.2

Let $x(t)$, $t \in (t_0, T)$, $(0 \leq t_0 < T < +\infty)$ be a non-extendable to the right solution of (1.1). Then

$$\lim_{t \to T^-} x(t) = +\infty$$

Theorem 3.3

Let V: $R_+ \times R^n \rightarrow R$ be a Lyapunov function satisfying

$$V(t,u) \leq (t,u), (t,u) \in \mathbb{R}_+ \times \mathbb{R}^n \tag{3.1}$$

and $V(t, u) \rightarrow +\infty$ as $u \rightarrow R$

Uniformly with respect to t lying in any compact set. Here, Y: $R_+ \times R$ is continuous and such that for every $(t_0, u_0) \in R_+ \times R$, the problem

$U = Y(t, u) + \epsilon,\ u(t_0) = u_0 + \epsilon$ (3.2)

Has a maximal solution defined on $(t_{0v}, +\infty)$. Then, every solution of (1.1) is extendable to $+\infty$.

Proof: Let (t_0, T) be the maximal interval of existence of solution $x(t)$ of (1.1) and assume that $T < +\infty$. Let $y(t)$ be the maximal solution of (3.1) with $y(t_0) = V(t_0, x(t_0))$. Then, since

Du $(t) \leq Y(t, v(t)), t \in (t_0, t_1+\alpha)\backslash S$: s–a countable set (3.3)

We have

$V(t, x(t)) \leq Y(t), t \in [t_0, T]$ (3.4)

On the other hand, since $x(t)$ is non-extendable to the right solution and we have

$$\lim_{t \to r} x(t) = +\infty$$

This implies that $V(t)$ converges to $+\infty$ as $t \to T$ but (3.4) implies that

Lim Sup $V(t, x(r)) \leq Y(T)$

As $t \to T$ thus $T = +\infty$

Corollary 3.4

Assume that there exists $\alpha > 0$ such that

$\|F(t, u)\| \leq Y \in R, \|u\| > \alpha$

Where $Y: R_+ \times R_+ \to R_+$ is such that for every $(t_0, u_0) \in R_+$ the problem (3.2) has a maximal solution defined on (t_0, ∞). Then, every solution of (1.1) is extendable to $+\infty$

Proof: Here, it suffices to take $V(t, u) = u$ and obtain

$$V_\epsilon(t, x(t)) = \lim_{h \to 0} \frac{\|x(t) + hF(t, x(t)) - x(t)\|}{h}$$

$$\leq \|F(t, x(t))\| \leq Y(t, V(t, x(t)))$$

Provided that $\|x(t)\| > \alpha$.

Now, let $x(t)$, $t \in (t_0, T)$ be a solution non continuable to the right solution of (1.1) such that $R < +\infty$. Then, for t sufficiently close to T from left, we have $\|x(t)\| > \alpha$. However, if $Y: R_+ \times R \to R$ be continuous and $(t_0, x_0) \in R_+ \times R$ $\alpha \in (0, +\infty)$ be the maximal solution of (3.2) in the interval $(t_0, t_0 + \alpha)$, let $V: (t_0, t_0 + \alpha) \to R$ be continuous and such that $u(t_0) \leq u_0$ and DU $(t) \leq Y(t, u(t)), t \in (t_0, t_0 + \alpha)\backslash S$ and S is a countable set then

$u(t) \leq s(t), t \in (t_0, t_0 + \alpha)$

This implies that every solution $x(t)$ of (1.1) is extendable and hence the proof.

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