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**RESEARCH ARTICLE**

# ON SERIES OF INEQUALITIES VIA VARIOUS ITERATION SCHEMES WITH SELF AND CONTRACTION MAPPINGS IN BANACH SPACE UNDER LIMITING CONDITIONS

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## ABSTRACT

Through iterative procedures, our aim is to connect the different inequalities and fixed-point issues arising from self, contractive and non-expansive mappings in Banach spaces in this communication. We offer an iterative technique for resolving the fixed-point issues and various inequalities under study. We demonstrate how well the suggested approach converges.

**Keywords**:Non-expansive mapping,Continuous mappings,Self mappings, Banach spaces,Fixed point theory etc.

## INTRODUCTION

Let $T$ be the self-map defined on $X$ in the metric space $(X, D)$. Making the premise that the set of fixed points for $T$ is represented by $F(T) = \{z \in X: Tz = z\}$. The sequence $\{x_n\}_{n=0}^{\infty}$for $x_0 \in X$. The Picard iteration, defined as $x_{n+1} \in Tx_n, n \geq 0$, is used in mathematics. The sequence $\{x_n\}_{n=0}^{\infty}$ defines $x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, n \geq 0$ for the value of $\{\alpha_n\}_{n=0}^{\infty}$.This sequence appears in $(0,1)$. The Mann iteration process [6] is denoted by the notation $\sum_{n=0}^{\infty} \alpha_n = \infty$. In addition to studying iteration and fixed point non-expansive mapping in Banach space in 1976, Ishikawa [4, 5] discovered fixed points using a new iteration method.

In 2000, Noor [7] introduced the following iteration scheme for arbitrary chosen $x_1 \in C$ construct the sequence $\{x_n\}$ by

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T z_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n T x_n \end{aligned}\right\}$$

For all $n \geq 1$Where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$.

Later, in 2014, Abbas et al. [1] offered the iteration below, where a sequence $\{x_n\}$ is created from randomly selected $x_1 \in C$ by

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)Ty_n + \alpha_n Tz_n \\ y_n &= (1-\beta_n)Tx_n + \beta_n Tz_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n Tx_n \end{aligned}\right\}$$

**Definition 1.1** Let $H$ be a non-empty subset of $X$, a Banach space. Let $T$ once more be the self-map established on $X$. Consequently, $T$ is said to mean non-expansive if $\|Tu - Tv\| \le p\|u - v\| + q\|u - Tv\| \ \forall\, u, v \in H$and $p, q : p + q \le 1$. The inverse of this relation, that is, that a mean non-expansive mapping may not be a non-expansive mapping, is often untrue. Every non-expansive mapping is a mean non-expansive mapping with $p = 1$and $q = 0$. We have thought about the generalized version of mean non-expansive mapping by taking into account $\|Tu - Tv\| \le p\|u - v\| + q\|u - Tv\| \ \forall\, u, v \in H$and $p, q : p + q < 1$.

**Definition 1.2** For some initial approximation $x_0 \in H$ consider the following sequence

$$\left.\begin{aligned} x_{n+1} &= T\left(\frac{x_n + y_n}{2}\right), \\ y_n &= (1-\alpha_n)x_n + \alpha_n T\left(\frac{x_n + y_n}{2}\right), \end{aligned}\right\}$$

$x_0$is the initial approximation such that $x_0 \in H$ and $\{\alpha_n\}_{n=0}^{\infty} \in [0,1]$.

**Definition 1.2** For some initial approximation $x_0 \in H$ consider the following sequence

$$\left.\begin{aligned} x_{n+1} &= T\left(\frac{x_n + y_n}{2}\right), \\ y_n &= (1-\delta)x_n + \delta T\left(\frac{x_n + y_n}{2}\right), \end{aligned}\right\}$$

$x_0$is the initial approximation such that $x_0 \in H$ and $\delta \in [0,1]$. The definitions of the rate of convergence that follow are credited to Berinde [2].

**Definition 1.3** Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of real numbers converging to $\alpha$ and $\beta$ respectively. If $\lim_{n\to\infty} \left\|\frac{\alpha_n - \alpha}{\beta_n - \beta}\right\| = 0$, then $\{\alpha_n\}$ converges faster than $\{\beta_n\}$.

**Definition 1.4** Suppose that for two fixed-point iteration processes $\{u_n\}$and $\{v_n\}$, both converging to the same fixed point $w$, the error estimates $\|u_n - w\| \le p_n$ and $\|v_n - w\| \le q_n$ for all $n \ge 1$, are available where $\{p_n\}$ and $\{q_n\}$ are two sequences of positive numbers converging to zero. If $\{p_n\}$ converges faster than $\{q_n\}$, then $\{u_n\}$ converges faster than $\{v_n\}$ to $w$.

**Lemma 1.5 [3]** Let $C$ be a non-empty closed convex subset of a uniformly convex Banach space $E$, and $T$ a non-expansive mapping on $C$. Then, $1 - T$ is demiclosed at zero.

**Lemma 1.6 [8]** Suppose $C$ be a uniformly convex Banach space and $0 < p \le t_k \le q < 1$ for all $n \in N$. Let $\{u_k\}$ and $\{v_k\}$ be two sequences of $C$ such that $\limsup_{k\to\infty} \|u_k\| \le r$ also we have $\limsup_{k\to\infty} \|v_k\| \le r$ and $\limsup_{k\to\infty} \|t_k u_k + (1 - t_k)v_k\| = r$ holds for some $r \ge 0$. Then, $\lim_{k\to\infty} \|u_k - v_k\| = 0$.

**RESULTS**

**Theorem 2.1** If $K$ be any non-empty subset of a Banach space $X$ and $T$ be the self-map on $K$ satisfying the non-linear $\|Tu - Tv\| \leq \|u - v\| - m\|x - Ty\|$ and iterative scheme for the sequence $\{u_r\}_{r=0}^{\infty}$ given by $w_r = (1 - \tau_r)u_r + \tau_r Tu_r,\ v_r = Tw_r$ also $u_{r+1} = Tv_r$ with $0 < \{u_r\} \leq 1$ and $\sum_{r=0}^{\infty} \tau_r = \infty$. Then show that the inequality

$$\|u_{r+1} - s\| \leq (1 - m)^{2(r+1)}\|u_0 - s\|\prod_{k=0}^{n}(1 - m\tau_0)$$

**Proof:** Assume that $s \in F(T)$. So, from the given criterian we get

$$\|w_r - s\| = \|(1 - \tau_r)u_r + \tau_r Tu_r - s\|$$

$$\leq (1 - \tau_r)\|u_r - s\| + \tau_r\|Tu_r - s\|$$

$$\leq (1 - \tau_r)\|u_r - s\| + \tau_r\|u_r - s\| - m\|u_r - Ts\|$$

$$\leq (1 - \tau_r + \tau_r - \tau_r s)\|u_r - Ts\|$$

$$\|w_r - s\| \leq (1 - \tau_r s)\|u_r - s\|$$

Also, $$\|v_r - s\| = \|Tw_r - s\|$$

$$i.e. \|v_r - s\| \leq (1 - s)\|w_r - s\|$$

Hence, from the above two inequalities we achieve

$$\|v_r - s\| = (1 - m\tau_r)(1 - m)\|u_r - s\|$$

Therefore, $$\|u_{r+1} - s\| = \|Tv_r - s\|$$

$$i.e. \|u_{r+1} - s\| \leq (1 - m)\|Tv_r - s\|$$

From the above two inequalities, we achieve

$$\|u_{r+1} - s\| \leq (1 - m)^2(1 - m\tau_r)\|u_r - s\|$$

Hence, from the above two inequality we estimate

$$\|u_{r+1} - s\| \leq (1 - m)^2(1 - m\tau_r)\|u_r - s\|$$

$$\|u_r - s\| \leq (1 - m)^2(1 - m\tau_{r-1})\|u_{r-1} - s\|$$

$\|u_{r-1} - s\| \leq (1 - m)^2(1 - m\tau_{r-2})\|u_{r-2} - s\|$………by applying similar argument we achieve

$$\|u_1 - s\| \leq (1 - m)^2(1 - m\tau_0)\|u_0 - s\|$$

Thus, $$\|u_{r+1} - s\| \leq (1 - m)^{2(r+1)}\|u_0 - s\|\prod_{k=0}^{n}(1 - m\tau_0)$$

Hence, the required inequality.

**Limiting case:** But, $\tau_r \in [0,1]\ \forall\, r \in N, m \in [0,1)$. Now, applying the limiting criteria $n$ approaches to $\infty$. We achieve $\lim_{r\to\infty}\|u_{r+1} - s\| = 0$, from the above inequality and hence, $\{u_r\}_{r=0}^{\infty}$ converges to a fixed point $s$ of $T$.

**Theorem 2.2** Let $K$ be a closed, convex subset of a real normed linear space $X$ and $T$ be a self and contraction mapping on $K$ satisfying the criterion $\|Tu - Tv\| \leq \frac{\psi\|u-Tu\|+b\|u-v\|}{1+k\|u-Tu\|}$. Let $\{u_r\}_{r=0}^{\infty}$ be the sequence generated by the iterative processes

$$\left.\begin{aligned} u_{r+1} &= T\left(\tfrac{u_r+v_r}{2}\right), \\ v_r &= (1-\tau_r)u_n + \tau_r T\left(\tfrac{u_r+v_r}{2}\right), \end{aligned}\right\}$$

$u_0$is the initial approximation such that $u_0 \in K$ and $\{\tau_r\}_{r=0}^{\infty} \in [0,1]$. Also,

$$\left.\begin{aligned} u_{r+1} &= T\left(\tfrac{u_r+v_r}{2}\right), \\ v_r &= (1-\delta)u_r + \delta T\left(\tfrac{u_r+v_r}{2}\right), \end{aligned}\right\}$$

$u_0$is the initial approximation such that $u_0 \in K$ and $\delta \in [0,1]$

respectively with sequence $\{w_r\}_{r=0}^{\infty} \in [0,1]$. Then show that the inequality

$$\|u_{r+1} - s\| \leq \left(\frac{\rho}{2}\right)^{r+1} \|u_r - s\| \prod_{i=0}^{r+1} \left\{1 + \frac{1 - \tau_r + \tau_r \frac{\rho}{2}}{1 - \tau_r \frac{\rho}{2}}\right\}$$

**Proof:** Suppose that$s$ be the fixed point of the mapping $T$. Then by using the first iterative process, we have

$$\|u_r - s\| = \left\|\frac{u_r + v_r}{2} - s\right\| \leq \left\|\frac{u_r + v_r}{2} - s\right\| \leq \frac{\rho}{2}\|u_r - s\| + \frac{\rho}{2}\|v_r - s\|$$

Now, $\|v_r - s\| = \left\|(1 - w_r)u_r + \tau_r T\left(\frac{u_r+v_r}{2}\right) - s\right\|$

$$\leq (1 - \tau_r)\|u_r - s\| + \tau_r \left\|T\left(\frac{u_r + v_r}{2}\right) - s\right\|$$

$$\leq (1 - \tau_r)\|u_r - s\| + \tau_r \rho \left\|T\left(\frac{u_r + v_r}{2}\right) - s\right\|$$

$$\leq (1 - \tau_r)\|u_r - s\| + \tau_r \frac{\rho}{2}\|u_r - s\| + \tau_r \frac{\rho}{2}\|v_r - s\|$$

$$i.e. \left(1 - \tau_r \frac{\rho}{2}\right)\|v_r - s\| \leq \|u_r - s\| + \tau_r\|u_r - s\| + \tau_r \frac{\rho}{2}\|u_r - s\|$$

$$i.e. \|u_{r+1} - s\| \leq \frac{\rho}{2}\left\{1 + \frac{1 - \tau_r + \tau_r \frac{\rho}{2}}{1 - \tau_r \frac{\rho}{2}}\right\}\|u_r - s\|$$

in the same manner we can claim $\|u_r - s\| \leq \frac{\rho}{2}\left\{1 + \frac{1-\tau_r+\tau_r\frac{\rho}{2}}{1-\tau_r\frac{\rho}{2}}\right\}\|u_{r-1} - s\|$ ... and hence the last normed linear factor will be $\|u_1 - s\|$ and which is less than or equal to $\frac{\rho}{2}\left\{1 + \frac{1-\tau_r+\tau_r\frac{\rho}{2}}{1-\tau_r\frac{\rho}{2}}\right\}\|u_0 - s\|$. combining all inequalities, we get $\|u_{r+1} - s\| \leq \left(\frac{\rho}{2}\right)^{r+1} \|u_r - s\|\prod_{i=0}^{r+1}\left\{1 + \frac{1-\tau_r+\tau_r\frac{\rho}{2}}{1-\tau_r\frac{\rho}{2}}\right\}$ .This completes the proof.

**Limiting case:** Now, Applying the limiting criteria $n$ approaches to $\infty$. We achieve $\lim_{r\to\infty}\|u_{r+1} - s\| = 0$, from the above inequality and hence, $\{u_r\}_{r=0}^{\infty}$ converges to a fixed point $s$ of $T$.

**Example 2.3** Assuming $T(u) = \frac{u}{2}$, let $K$ and $T: K \to K$ be a contraction mapping. Consider the following iteration methods with the initial approximations $u_0 = 0.1$and $\{\tau_r\} = \frac{1}{2}$ :

$$\left.\begin{aligned} u_{r+1} &= T\left(\frac{u_r+v_r}{2}\right), \\ v_r &= (1-\tau_r)u_n + \tau_r T\left(\frac{u_r+v_r}{2}\right), \end{aligned}\right\}$$ $u_0$is the initial approximation such that $u_0 \in K$ and $\{\tau_r\}_{r=0}^{\infty} \in [0, 1]$. Also,

$$\left.\begin{aligned} u_{r+1} &= T\left(\frac{u_r+v_r}{2}\right), \\ v_r &= (1-\delta)u_r + \delta T\left(\frac{u_r+v_r}{2}\right), \end{aligned}\right\}$$ $u_0$is the initial approximation such that $u_0 \in K$ and $\delta \in [0, 1]$

respectively with sequence $\{w_r\}_{r=0}^{\infty} \in [0, 1]$.We notice that, for both iterative techniques, $\{u_r\}$ converges at zero in the $28^{th}$ approximation, indicating an equivalent rate of convergence.

**Theorem 2.4**Let $K$ be a non-empty, closed and convex subset of uniform convex Banach space (UCBS) $X$. Also $T$ be a non-expansive self mapping on $K$ and $\{u_r\}$ be a sequence defined such that

$$\left.\begin{aligned} u_{r+1} &= (1-\theta_r)Tv_r + \theta_r Tw_r \\ v_r &= (1-\varphi_r)w_r + \varphi_r Tw_r \\ w_r &= (1-\omega_r)u_r + \omega_r Tu_r \end{aligned}\right\}$$ where $\{\theta_r\}, \{\varphi_r\}$ and $\{\omega_r\}$ are real sequence in $(0, 1)$. Also, $F(T) \neq \emptyset$.

Then show that the inequality $\|S_{n,m}x - S_{n,m}y\| \leq \left[\prod L_{j}{}_{j=n}^{n+m-1}\right] [\|x - y\| + \sum_{i=n}^{n+m-1} \rho_i] \forall\, x, y \in C$.

**Proof:** Letting, $\lim_{r\to\infty}\|u_r - s\| = c$and $\limsup_{r\to\infty}\|v_r - s\| \leq c$, $\limsup_{r\to\infty}\|w_r - s\| \leq c$. Here, $T$ be a non-expansive self-mapping on $K$. So, $\|Tu_r - s\| \leq \|u_r - s\|$, $\|Tv_r - s\| \leq \|u_r - s\|$, and $\|Tw_r - s\| \leq \|u_r - s\|$. Taking limsup on both sides, we achieve the results $\limsup_{r\to\infty}\|Tu_r - s\| \leq c$, $\limsup_{r\to\infty}\|Tv_r - s\| \leq c$, and $\limsup_{r\to\infty}\|Tw_r - s\| \leq c$.

Since $c = \lim_{r\to\infty}\|u_{r+1} - s\| = \lim_{r\to\infty}\|(1-\theta_r)Tv_r + \theta_r Tw_r - s\|$

Ofcourse, we can modify the iteration scheme

$$c = \lim_{r\to\infty}\|u_{r+1} - s\| = \lim_{r\to\infty}\|(1-\theta_r)Tw_r + \theta_r Tv_r - s\|$$

$$\leq \lim_{r\to\infty} \|(1-\theta_r)(Tw_r - s) + \theta_r(Tv_r - s)\|$$

$$\leq \lim_{r\to\infty} [(1-\theta_r)\|Tw_r - s\| + \theta_r\|(Tv_r - s)\|]$$

$$\leq \lim_{r\to\infty} [(1-\theta_r)\|Tu_r - s\| + \theta_r\|(Tu_r - s)\|]$$

$$= \lim_{r\to\infty} \|Tu_r - s\|$$

But, $\lim_{r\to\infty} \|Tw_r - Tv_r\| = 0$

Now, $\|u_{r+1} - s\| = \|(1-\theta_r)Tw_r + \theta_r Tv_r - s\| \leq \|Tw_r - s\| + \theta_r\|Tw_r - Tv_r\|$

Hence, $c \leq \lim_{n\to\infty} \inf\|Tw_r - s\|$

Thus, $\lim_{n\to\infty} \|Tw_r - s\| = c$

On the other hand, we have

$$\|Tw_r - s\| \leq \|Tw_r - Tv_r\| + \|Tv_r - s\| \leq \|Tw_r - Tv_r\| + \|v_r - s\|$$

and this gives us $c \leq \lim_{n\to\infty} \inf\|v_r - s\|$

$$\lim_{n\to\infty} \|v_r - s\| = c$$

Using lemma 1.6, we get $\lim_{n\to\infty} \|w_r - Tw_r\| = 0$

Since, $\|v_r - s\| \leq \|w_r - s\| + \varphi_r\|Tw_r - w_r\|$

we write, $c \leq \lim_{n\to\infty} \sup\|w_r - s\|$

then, $\|w_r - s\| = c$ so, $c = \lim_{n\to\infty} \|w_r - s\|$

$$= \lim_{n\to\infty} \|(1-\theta_r)u_r + \theta_r Tu_r - s\|$$

$$= \lim_{n\to\infty} \|(1-\theta_r)(u_r - s) + \theta_r(Tu_r - s)\|$$

Now, setting $a_r(t) = \|tu_r + (1-t)v_1 - v_2\|$, $n \in N$ then $a_r(0) = \lim_{r\to\infty} \|v_1 - v_2\|$ and $a_r(1) = \lim_{r\to\infty} \|u_r - v_2\|$ exists. Hence, it is sufficient to show that the above expression is true for $t \in (0,1)$.

Taking $S_{n,m} = G_{n,m}G_{n+m-2} \ldots G_n \; \forall \, n, m \in N$. Then, $u_{n+m} = S_{n,m}u_{n,}, S_{n,m}v = v \; \forall \; \cap_{n\in N} F(G_n)$ and $\|S_{n,m}u - S_{n,m}v\| \leq \left[\prod L_j{}_{j=n}^{n+m-1}\right] [\|u - v\| + \sum_{i=n}^{n+m-1} \rho_i] \forall \, u, v \in K$ andthis is our desired inequality.

**Limiting case:** and by Lemma 1.6, we achieve $\lim_{r\to\infty} \|u_r - Tu_r\| = 0$.

**Example 2.5** Suppose $K = [1, 50]$and $X = R$. Let $T: K \to K$ be a mapping with the definition given by $T(u) = \sqrt{u^2 - 9u + 54}$ for all $u \in K$. Select $\theta_r = \varphi_r = \omega_r = \frac{3}{4}$, with $u_1 = 30$ as the beginning value. Then, using the aforementionediteration methods, we see that, in the $41^{st}$ approximation, $\{u_r\}$ converges at 6, for both iterative schemes, indicating an identical rate of convergence.

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