AJMS
Asian Journal of Mathematical Sciences

RESEARCH ARTICLE

TRANSCENDENTAL CANTOR SETS AND TRANSCENDENTAL CANTOR FUNCTIONS

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Received: 22-09-2023; Revised: 30-10-2023; Accepted: 16-11-2023

ABSTRACT

In this article, we consider the self-similar generalized Cantor set $C^n(\{i\}) = C^n(\{i_1,,,i_l\})$, and we establish the existence of probability true measure μ such that $\mu(E) = \frac{1}{s}\sum_{j=0,..,s-1}\mu\left(\varphi_j^{-1}(E)\right)$ generated by $C^n(\{i\})$. The Holder order α of the set $C^n(\{i\})$ is $\log_n(s)$ and we establish that $\left(\overline{\theta}^{\alpha}(\mu,x)\right)^{-\frac{1}{\alpha}} + n^2\left(\underline{\theta}^{\alpha}(\mu,x)\right)^{-\frac{1}{\alpha}} = s(i_l + i_l n)$ for all not finite n-adic $x \in C^n(\{i_1,...,i_l\})$. Transcendental numbers, such as e and π are a mathematical expression of nature, we introduce the transcendental Cantor set generated by transcendental numbers, which can be defined by $C^n(\xi) = \lim_{k\to\infty} C_k = \bigcap_{k=0,1,...,} C_k$, where the sequence $\{C_k\}$ is non-increasing and corresponds with the transcendental number ξ, for such a set, we consider an analog of the Cantor function.

Keywords: transcendental number, Cantor set, Cantor function, fractal, irregular Cantor set, Holder continuity.

INTRODUCTION

In recent years there have been several variants of generalization of the Cantor sets [4, 16], the best-known example of such generalization of the Cantor ternary set is the Smith-Volterra-Cantor sets [4], which presents the nowhere-dense self-similar set with positive Lebesgue measure. All iterations in the construction of the Smith-Volterra-Cantor set are self-similar, namely, each step generates the next steps so that each subinterval is divided in the same ratio. A singular function constructed on the Smith-Volterra-Cantor set, similar to the Cantor function, is Lipshitz on its domain.

In this paper, we develop two ways of generalization of self-similar sets, the first is we consider Cantors sets $C^n(\{i\}) = C^n(\{i_1,,,i_l\})$ with an arbitrary base, for such sets we establish that there exists a uniquely defined probabilistic measure μ defined by $\mu(E) = \frac{1}{s}\sum_{j=0,..,s-1}\mu(\varphi_j^{-1}(E))$ for all Borel sets E,and consider point wise densities for their measure generalized classical results of De-Jun Feng; the second we construct the irregular or transcendental Cantor set generated by the number e, which we denote by $C^3(10^{-2}e)$ such set can be presented as $C^3(10^{-2}e) = \lim_{k\to\infty} C_k = \bigcap_{k=0,1,...,} C_k$, where the collection $\{C_k\}$ is a sequence of non-increasing sets corresponded to the number e. For the irregular Cantor sets, we construct the functions analogous to Smith-Volterra-Cantor functions.

GENERALIZED CANTOR FUNCTION $G:[0,1]\to[0,1]$

Let $x \in [0,1] \subset R$ then for each natural number n there exists a unique expansion of x in the form of an infinite series

$$x = \sum_{k=1,...,} \frac{a_k(x)}{n^k} = \frac{a_1}{n^1} + \frac{a_2}{n^2} + \frac{a_3}{n^3} + \frac{a_4}{n^4} + ...$$

Where $\{a_k(x)\}_{k=1,...}$ with $a_k(x) \in \{0,1,...,n-1\}$

Definition 1 The generalized Cantor set $C^n(\{i\}) = C^n(\{i_1,,,i_l\})$ consists of all real numbers $x \in [0,1]$, which remain after the removal of all open intervals

$$\left(\frac{i_1}{n},\frac{i_1+1}{n}\right)\cup....\cup\left(\frac{i_l}{n},\frac{i_l+1}{n}\right)\cup\left(\frac{i_1}{n^2},\frac{i_1+1}{n^2}\right)\cup...\cup\left(\frac{i_l}{n^2},\frac{i_l+1}{n^2}\right)......$$

Or the generalized Cantor set $C^n(\{i\})$ consists of all real numbers $x \in [0,1]$ the n- expansion of which can be written $x = \sum_{k=1,...,} \frac{a_k(x)}{n^k}$ without a set $\{i_1,,,i_l\}$ of numbers $i_j \in \{0,1,...,n-1\}$, $j = 1,...,l$.

Let $x \in [0,1]$ be expressed in base n, we define the Cantor function $G:[0,1]\to[0,1]$ by

$$G(x) = \sum_{k=1,...,} \frac{b_k(x)}{s^k}, \quad b_k(x) \in \{0,1,...,s-1\}$$

for all $x \in C^n(\{i\}) = C^n(\{i_1,,,i_l\})$, where $b_k(x) = j(a_k)$ defined by

$$\{0,1,...,i_1-1, i_1+1,...., i_l-1, i_l+1,..., n-1\} \to \{0,1,..., s-1\}$$

for all $a_k(x)$ such that $x \in C^n(\{i\})$, and

$$G(x)=\sup_{y\le x}\left\{G(y),\quad y\in C^n(\{i\})\right\}$$

For all $x\in[0,1]\setminus C^n(\{i\})$ this definition is correct since $s=n-l$.

Theorem1. For all $x, y\in[0,1]$, the Holder condition for the generalized Cantor function G given by

$$\left|G(x)-G(y)\right|\le c\left|x-y\right|^{\alpha}$$

Holds with the best possible constant $\alpha=\log_n(s)$

THE CUMULATIVE PROBABILITY DISTRIBUTION FUNCTION

For a given general Cantor set $C^n(\{i\})$, we define a set $\{\varphi_j,\quad j=0,...,s-1\}$ of functions defined by

$$\varphi_0=\frac{1}{n}x,$$

$$\varphi_1=\frac{i_1+1}{n}+\frac{1}{n}x,$$

$$\ldots\ldots,$$

$$\varphi_{s-1}=\frac{i_l+1}{n}+\frac{1}{n}x$$

Defined for all $x\in R$

Example In cases $C^3(\{0\})$ and $C^3(\{2\})$, we have

$\varphi_0=\frac{1}{3}+\frac{1}{3}x,\qquad \varphi_1=\frac{2}{3}+\frac{1}{3}x$

and

$$\varphi_0=\frac{1}{3}x,\qquad \varphi_1=\frac{1}{3}+\frac{1}{3}x.$$

Theorem2.For any given general Cantor set $C^n(\{i_1,...,i_l\})$ there exists a uniquely defined probability true measure μ such that

$$\mu(E)=\frac{1}{s}\sum_{j=0,..,s-1}\mu\left(\varphi_j^{-1}(E)\right)$$

for all Borel sets E. For all continuous functions $g:R\to R$, the following equality

$$\int_E g(x)\,d\mu(x)=\frac{1}{s}\sum_{j=0,..,s-1}\int_E g\left(\varphi_j(x)\right)d\mu(x)$$

For all Borel sets E.

The proof of this theorem is based on the idea that, for all Borel sets $E \subset R$ and any measure $\eta \in \mathrm{M}$, since mapping $\Upsilon : \mathrm{M} \to \mathrm{M}$ given by

$$\Upsilon(\eta)(E) = \frac{1}{s} \sum_{j=0,..,s-1} \mu\left(\varphi_j^{-1}(E)\right)$$

is contractive transformation on M, the mapping $\Upsilon : \mathrm{M} \to \mathrm{M}$ has a unique fixed point, namely, the measure $\mu \in \mathrm{M}$ such that $\Upsilon(\mu) = \mu$, such measure μ satisfies the identity

$$\int_E g(x)\, d\mu(x) = \frac{1}{s} \sum_{j=0,..,s-1} \int_E g\left(\varphi_j(x)\right) d\mu(x)$$

for all continuous functions $g : R \to R$.

LOWER AND UPPER DENSITIES OF THE GENERALIZED CANTOR FUNCTION

The lower and upper γ-densities of the measure μ at a point x is defined by

$$\overline{\theta}^{\gamma}(\mu, x) = \liminf_{h\to 0} \frac{\tilde{G}(x+h) - \tilde{G}(x-h)}{|2h|^{\gamma}}$$

And

$$\underline{\theta}^{\gamma}(\mu, x) = \limsup_{h\to 0} \frac{\tilde{G}(x+h) - \tilde{G}(x-h)}{|2h|^{\gamma}},$$

Where the function $\tilde{G}(x)$ is an extension of the generalized Cantor function given by

$$\tilde{G}(x) = \begin{cases} 0, & x < 0 \\ G(x), & 0 \le x \le 1 \\ 1, & 1 < x. \end{cases}$$

The common value $\overline{\theta}^{\gamma}(\mu, x) = \underline{\theta}^{\gamma}(\mu, x)$, when such exists, is called the γ-density of the measure μ at x.

Definition 2. For $x \in [0, 1]$, we define a pair of functions $\tilde{\tau}(x)$ and $\tau(x)$ by

$$\tilde{\tau}(x) = \liminf_{j\to\infty} \sum_{k=1,\dots,} \frac{a_{k+j}(x)}{n^k}$$

and

$$\tau(x) = \min\left\{\tilde{\tau}(x), \tilde{\tau}(1-x)\right\}.$$

Definition 3. We introduce transformation

$$\Phi : C^n\left(\{i_1,\dots,i_l\}\right) \to C^n\left(\{i_1,\dots,i_l\}\right)$$

by

$$\Phi(x)=\begin{cases} n, & x\in\left[0,n^{-1}\right],\\ n\left(x-\frac{1}{n}\right), & x\in\left[\frac{1}{n},\frac{2}{n}\right],\\ \dots\dots\dots\dots \\ n\left(x-\frac{i_j-1}{n}\right), & x\in\left[\frac{i_j-1}{n},\frac{i_j}{n}\right],\\ n\left(x-\frac{i_j+1}{n}\right), & x\in\left[\frac{i_j+1}{n},\frac{i_j+2}{n}\right],\\ \dots\dots, \\ n\left(x-\frac{i_l+1}{n}\right), & x\in\left[\frac{i_l+1}{n},1\right]. \end{cases}$$

for all $x=\sum_{k=1,\dots,} a_k(x)n^{-k}$ **,** $a_k\in\left\{\{0,1,\dots,n-1\}\setminus\{i_1,,,i_l\}\right\}$**.**

Theorem 3. Let $C^n(\{i\})$ **be generalized Cantor set and let** $x\in C^n(\{i\})$ **then**

$$\overline{\theta}^{\alpha}(x)=\left(si_l-sn\tau(x)\right)^{-\alpha}$$

and

$$\underline{\theta}^{\alpha}(\mu,x)=\begin{cases} s^{-\alpha} & x \text{ is a n-finite}\\ \left(\frac{si_l+s\tau(x)}{n}\right)^{-\alpha} & \text{otherwise}; \end{cases}$$

For all not finite n**- adic** $x\in C^n\left(\{i_1,\dots,i_l\}\right)$ **we have**

$$\left(\overline{\theta}^{\alpha}(\mu,x)\right)^{-\frac{1}{\alpha}}+n^2\left(\underline{\theta}^{\alpha}(\mu,x)\right)^{-\frac{1}{\alpha}}=s\left(i_l+i_l n\right).$$

Proof.

First, we show that assume $x\in\left[0,n^{-1}\right]$ and $\max\left\{x,n^{-1}-x\right\}\le t\le 1-x$ then we obtain

$$\mu\left(\left[x-t,x+t\right]\right)\ge (st)^{\alpha}\left(si_l-snx\right)^{-\alpha}.$$

Indeed, if $\max\left\{x,n^{-1}-x\right\}\le t\le \frac{i_l}{n}-x$ then $\left[0,n^{-1}\right]\subset\left[x-t,x+t\right]$

so

$$\frac{\mu([x-t, x+t])}{(st)^\alpha} \geq \frac{1}{s(st)^\alpha} \geq (si_l - snx)^{-\alpha};$$

if $\frac{i_l}{n} - x \leq r \leq 1-x$ then

$$\frac{\mu([x-t, x+t])}{(st)^\alpha} \geq \frac{\frac{s-1}{s}+s^{-\alpha}r^\alpha}{s^\alpha\left(\frac{s-1}{s}-x+r\right)^\alpha} > \frac{\frac{s-1}{s}}{s^\alpha\left(\frac{s-1}{s}-x\right)^\alpha} \geq (si_l - snx)^{-\alpha},$$

where we used the following estimation

$$\frac{t^\alpha}{s} \leq \mu([0, t]) \leq t^\alpha$$

holds for each $t \in [0,1]$.

Therefore, for each $x \in C^n(\{i\})$ and all $t \in (0, n^{-1})$, we can choose the sequence $\{j_d\}$ such that $x = \lim_{k\to\infty} \varphi_{j_1} \circ ... \circ \varphi_{j_k}([0,1])$, we take $y = \lim_{k\to\infty}(\varphi_{j_1} \circ ... \circ \varphi_{j_{k-1}})^{-1}(x)$ so that $y = (\varphi_{j_1} \circ ... \circ \varphi_{j_{k-1}})^{-1}(x)$ and $y = \Phi^{k-1}(x)$ where mapping Φ^{k-1} is the $k-1$-th iteration of mapping Φ. Therefore, we have

$$(\varphi_{j_1} \circ ... \circ \varphi_{j_{k-1}})^{-1}([x-t, x+t]) = [y-\tilde{t}, y+\tilde{t}]$$

for all $\tilde{t} = tn^{k-1}$. So, we have $\mu([x-t, x+t]) = n^{-(k-1)\alpha}\mu([y-\tilde{t}, y+\tilde{t}])$ hence

$$\mu([x-t, x+t]) = n^{-(k-1)\alpha}\mu([y-\tilde{t}, y+\tilde{t}]).$$

Since $y = \Phi^{k-1}(x)$ and $1-y = \Phi^{k-1}(1-x)$ we have

$$\liminf_{t\to 0} \frac{\mu([x-t, x+t])}{(st)^\alpha} \geq$$

$$\geq$$

$$\geq \left(si_l - sn\left(\min\left\{\liminf_{k\to\infty} \Phi^k(x), \liminf_{k\to\infty} \Phi^k(1-x)\right\}\right)\right)^{-\alpha} =$$

$$= (si_l - sn\tau(x))^{-\alpha},$$

that proves the theorem.

IRREGULAR OR TRANSSENDERNTAL CANTOR SETS

In this chapter, we introduce new class sets, which are uncountable, compact, perfect, and totally disconnected. Such sets are closely related to the Cantor and Smith-Volterra sets, however, their nature is completely different from the regular fractal sets, so we will call this class irregular fractals.

As direct corollaries of the Hermite-Weierstrass theorem, we obtain that the numbers e and π are transcendental, and we write expansions of e and π

$$10^{-2}e = 2\times10^{-2} + 7\times10^{-3} + 1\times10^{-4} + 8\times10^{-5} + 2\times10^{-6} + 8\times10^{-7} + 1\times10^{-8} + ...$$

and

$$10^{-2}\pi = 3\times10^{-2} + 1\times10^{-3} + 4\times10^{-4} + 1\times10^{-5} + 5\times10^{-6} + 9\times10^{-7} + 2\times10^{-8} + ...$$

By using the transcendental number e, we construct the irregular Cantor sets from the unit compact interval $C_0 = [0,1]$ by performing the recursive process: the first iteration consists of the removal of the open interval $\left(\frac{1}{2} - \frac{2}{2\times10^2}, \frac{1}{2} + \frac{2}{2\times10^2}\right)$ from the interval $[0,1]$ so that the remaining set is

$$C_1 = \left[0, \frac{1}{2} - \frac{2}{2\times10^2}\right] \cup \left[\frac{1}{2} + \frac{2}{2\times10^2}, 1\right];$$

the second iteration is removing the subintervals of the common width 7×10^{-3} from the middle of each of the two remaining intervals, so for the second step we leave the set

$$C_2 = \left[0, \frac{1}{2}\left(\frac{1}{2} - \frac{2}{2\times10^2}\right) - \frac{7}{4\times10^3}\right] \cup$$
$$\cup \left[\frac{1}{2}\left(\frac{1}{2} - \frac{2}{2\times10^2}\right) + \frac{7}{4\times10^3}, \frac{1}{2} - \frac{2}{2\times10^2}\right] \cup$$
$$\cup \left[\frac{1}{2} + \frac{2}{2\times10^2}, 1 - \frac{1}{2}\left(\frac{1}{2} + \frac{2}{2\times10^2}\right) - \frac{7}{4\times10^3}\right] \cup$$
$$\cup \left[1 - \frac{1}{2}\left(\frac{1}{2} + \frac{2}{2\times10^2}\right) + \frac{7}{4\times10^3}, 1\right];$$

employing expansions of the number e, we continue the process indefinitely and obtain a sequence $\{C_k\}$ of non-increasing sets C_k such that $C_k \supset C_{k+1}$, the set of points, that remain after infinite numbers of iterations, is called the irregular Cantor set generated by the number e and denoted by $C^3\left(10^{-2}e\right)$.

We can write $C^3\left(10^{-2}e\right) = \lim_{k\to\infty} C_k = \bigcap_{k=0,1,...,} C_k$.

Applying a similar process for the expansion of the number π, we obtain the irregular Cantor set generated by the number π and denoted by $C^3\left(10^{-2}\pi\right)$.

From the definitions, we have that

$$\mu_L\left(C^3\left(10^{-2}e\right)\right)=1-10^{-2}e$$

and

$$\mu_L\left(C^3\left(10^{-2}\pi\right)\right)=1-10^{-2}\pi .$$

The coefficient is a 10^{-2} is not important we can use any suitable coefficient, which guarantees the correctness of the iteration process.

The irregular Cantor sets $C^3\left(10^{-2}e\right)$ and $C^3\left(10^{-2}\pi\right)$ are similar to the ternary Cantor set in the sense that we are removing the part of the middle sets at each step of the iteration process, however, the nature of irregular sets is different from classical Cantor and Smith-Volterra-Cantor since they are regular in the sense that k-iteration depends systematic on $k-1$-iteration, on the contrary, the width of removed intervals in the irregular set is prescribed by the nature of the generated number is unregular.

The interior of the irregular Cantor set is empty, namely, it does not contain any interval, which is open in the standard topology of the real line. The irregular Cantor sets are not self-similar since, at each step of the iteration, the set of numbers fed by expansion

$$10^{-2}e = 2\times10^{-2} + 7\times10^{-3} + 1\times10^{-4} + 8\times10^{-5} + 2\times10^{-6} + 8\times10^{-7} + 1\times10^{-8} + \ldots$$ is unique.

PROPERTIES OF IRREGULAR CANTOR SETS AND IRREGULAR CANTOR FUNCTIONS

The irregular Cantor sets are uncountable, closed,and totally disconnected, with uniquely defined Lebesgue measure $\mu_L\left(C^n\left(\xi\right)\right)=10^{-2}\xi$ for the irregular Cantor set $C^n\left(\xi\right)$ generated by the number ξ, where the number n determines the base as was explained earlier. The proofs of these statements are similar to proofs for the classical Cantor set.

Based on the irregular Cantor set $C^3\left(10^{-2}e\right)$, we define the irregular Cantor function $F:[0,1]\to[0,1]$ by iteration procedure as follows:

$$F_1(x)=\begin{cases}0 & if \quad x\le 0,\\ \dfrac{1}{2} & if \quad x\in\left(\dfrac{1}{2}-\dfrac{2}{2\times10^{2}},\dfrac{1}{2}+\dfrac{2}{2\times10^{2}}\right),\\ 1 & if \quad 1\le x,\end{cases}$$

$$F_2(x)=\begin{cases}0 & if \quad x\le 0,\\ \frac{1}{4} & if \quad x\in\left(\frac{1}{2}\left(\frac{1}{2}-\frac{2}{2\times 10^2}\right)-\frac{7}{4\times 10^3},\frac{1}{2}\left(\frac{1}{2}-\frac{2}{2\times 10^2}\right)+\frac{7}{4\times 10^3}\right),\\ \frac{1}{2} & if \quad x\in\left(\frac{1}{2}-\frac{2}{2\times 10^2},\frac{1}{2}+\frac{2}{2\times 10^2}\right),\\ \frac{3}{4} & if \quad x\in\left(1-\frac{1}{2}\left(\frac{1}{2}+\frac{2}{2\times 10^2}\right)-\frac{7}{4\times 10^3},1-\frac{1}{2}\left(\frac{1}{2}+\frac{2}{2\times 10^2}\right)+\frac{7}{4\times 10^3}\right),\\ 1 & if \quad 1\le x,\end{cases}$$

et cetera for all $x\in[0,1]\setminus C^3(10^{-2}e)$, and, by continuity, we put function irregular Cantor function F linear on $C^3(10^{-2}e)$.

The irregular Cantor function $F:[0,1]\to[0,1]$ is a continuous monotone function, therefore, there exists the derivative F' of F for Lebesgue almost all $x\in[0,1]$. We have $F'(x)=0$ for all $[0,1]\setminus C^3(10^{-2}e)$ and $F'(x)=const$ for all $[0,1]\setminus\left(C^3(10^{-2}e)\cup\left\{\bigcup_{k=1,2,\ldots}\xi_k\right\}\cup\{0,1\}\right)$, where points ξ_k are points of division such as $\frac{1}{2}\left(\frac{1}{2}-\frac{2}{2\times 10^2}\right)-\frac{7}{4\times 10^3},\frac{1}{2}\left(\frac{1}{2}-\frac{2}{2\times 10^2}\right)+\frac{7}{4\times 10^3},\frac{1}{2}-\frac{2}{2\times 10^2},\ldots$. At each division point, there are two derivative numbers: one equals zero, and the second is a positive constant.

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