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RESEARCH ARTICLE

BIANCHI TYPE -II FRACTIONAL COSMOLOGICAL MODEL WITH VARIABLE LAMBDA

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ABSTRACT

In this paper, we have reviewed and presented the results of the fractional order differential field equation for a Bianchi type-II (LRS) space-time with a perfect fluid assuming a variable cosmological constant Λ. In order to substitute Einstein's field equations for a certain configuration with equivalent fractional field equations, we used the Last Step Modification (LSM) method. Further applied Riemann-Liouville fractional order derivative assuming lower terminal at 0 to obtain all physical properties of the developed model along with their graphical representation.

Keywords: Bianchi Type II Cosmological Model, Fractional Calculus, EOS Parameter, LSM Method.

INTRODUCTION

The universe is observed to be homogeneous and isotropic on large scales, but spatially homogeneous and anisotropic models are important for probing early-universe conditions, explaining the origin of isotropy, and examining possible deviations from the standard ΛCDM scenario. Such models have been extensively investigated within the framework of general relativity in the quest for an accurate representation of the early universe. Numerous dynamical spatially homogenous and anisotropic dark energy (DE) models of the Bianchi type models have been proposed in literature by different authors, both in GR and in modified theories of gravity. A dynamically anisotropic equation of state (EoS) parameter and a dynamical energy density in general relativity (GR) were used by Akarsu et al. [1] to obtain the anisotropy parameter's general form for the expansion of Bianchi type-III metric in the presence of a single imperfect fluid. They discovered that an accelerated expansion time period isotropizes both the anisotropy of the fluid and the expansion anisotropy. A host of authors have investigated spatially homogeneous and anisotropic DE models both in general relativity and in modified theories of gravitation (See [2], [3], [4]).

In the context of the gauge theory of gravity, Cervero [6] showed that a time-dependent cosmological constant emerges quite naturally as a result of Weyl conformal symmetry being broken by quantum processes. Which was a huge motivation to the researchers to use time-dependent cosmological constant. The Bianchi Type-II model's compatibility with a decay law for cosmological terms that was assumed to be proportional to $\ddot{a}/a$ was examined by Pradhan et al. [7]. Further Tiwari et al. [8] studied Bianchi type-II (LRS) space-time with a perfect fluid and a changeable cosmological constant. In order to solve the Einstein field equations, they assumed that the cosmological term Λ was proportional to R−m where m being a constant and R represents the scale factor. Similarly time-dependent gravitational and cosmological constants on Bianchi Type space – times investigated by several re-searchers assuming different conditions [9], [10], [11], [12], and [13].

In a variety of applications, fractional calculus has shown to accurately explain non-local space-time effects. Since there are numerous proposals to carry out the necessary generalisations, the incorporation of fractional derivatives in gravity is a challenging task. From the perspective of pure mathematics, fractional derivatives must be in the form of result in a fractional geometry such that the definitions of all the geometric components of general relativity, such as the connection, the Riemann tensor, the covariant derivative, and the metric, are presented in form of a fractional derivative. This method is called the First Step Modification (FSM). Modifying the Hilbert-Einstein field equations for a particular geometry by substituting the covariant derivative order by its equivalent fractional derivative,

without delving further into the process of how such an equation can be created, is a more practical and straightforward approach. This method is frequently referred to as a Last Step Modification (LSM). These two approaches were firstly introduced by Mark D. Roberts [14] in 2009.In the first approach the time derivative in the Friedman equation is replaced with a fractional time derivative using the Riemann-Liouville fractional derivative assuming lower terminal at 0. The Fractional Friedman Dynamical Equations developed by El-Nabulsi Ahmad Rami [15] in static universe. Although he was unable to solve the equations but his effort arise so many questions which attract the researchers to explore more. Standard Friedmann and scalar field equations including fractional derivatives of the scale factor were developed by V. K. Shchigolev [16]. He proposed several precise answers to those fractional cosmological model equations and also arise a question that "in what degree the ideas of the fractional differential calculus are productive in cosmology". He himself tried to find those unsolved questions by his work in [17], [18] and presented some solutions of time fractional cosmologies with different values of fractional parameter. The application of both techniques to gravity has recently been investigated at both the classical and cosmic levels. Formulating a variational principle for a fractional order action is a main strategy. Since a fractional variational theory is sufficiently generic for applications in other scientific fields, the scientific community, not just that of the gravitational region, is very interested in this latter approach. (see [19], [20] and [21]. Including the most recent work in 2021, V. K. Shchigolev [22] investigated fractional differential approach in cosmology in the frame of the accurate models of accelerated cosmological expansion. He described cosmological models of a scalar field with fractional derivatives, and Ernesto Barrientos et al. [23] presented an experimental model to extend the Friedmann equations with fractional derivatives by applying the Last Step Modification technique.

METHODOLOGY

The solution to optimal control problems involves the derivation of the optimal control characterization and the use of the forward and backward sweep for the state and control trajectories respectively[9]. By the Pontryagin's maximum, the optimal control can be analytically derived by the optimality condition while the state and control variables can be derived numerically using the Runge-Kutta of order six subject to the adjoint and transversality conditions[10]. In this brief review, we have presented a locally rotationally symmetrical Bianchi Type II cosmological model fractional differential with a perfect fluid and a variable cosmological constant Λ. We have selected to investigate the Bianchi type-II model that is locally rotationally symmetric and also evaluated the important features of the model. We have started with the assumption that the component of the shear tensor σi is proportional to the expansion scalar θ. Then applied the Last Step Modification (LSM) method, which substitutes similar fractional field equations for Einstein's field equations for a given configuration. In other words, $\delta_t \rightarrow D^\alpha$ after the field equations for a specific geometry have been derived. Further applied Riemann-Liouville fractional derivative with lower terminal at 0, after correcting a typo [24] (see page 310) is

$$D^\alpha_t t^m = \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} t^{m-\alpha}$$

The current manuscript is the revised version of our preprint [25]. The range of fractional parameters where the model demonstrates its existence and non-existence is included and the conclusion and discussion sections have been altered accordingly. Even if the physical relevance of this conclusion is currently unknown, it could potentially be useful for future research on fractional cosmologies.

Field Equation and Solution

We consider the metric for Bianchi type-II Cosmological Model given by:

$$(ds)^2 = \eta_{ij}\sigma^i\sigma^j, \tag{1}$$

where $\eta_{ij} = diag(-1, 1, 1, 1)$ and σi are given as $\sigma^0 = dt$, $\sigma^1 = A(t)\omega^1$, $\sigma^2 = B(t)\omega^2$, $\sigma^3 = B(t)\omega^3$, where $A(t)$, $B(t)$ are metric functions and ωi is defined as:

$$\omega^1 = dy + xdz, \quad \omega^2 = dz, \quad \omega^3 = dx,$$

energy momentum tensor

$$T_{ij} = (\rho + p)v_i v_j + pg_{ij},$$

choosen $v_i v^j = -1$, $p = \omega\rho$, $0 \le \omega \le 1$. Einstein's Field Equation with time-dependent Λ with 8πG = c = 1 are

$$R_{ij} - \frac{1}{2}R_k g_{ij} = -T_{ij} - \Lambda(t)g_{ij}$$

Gives

$$2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} - \frac{A^2}{4B^4} = \rho - \Lambda \tag{2}$$

$$2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} - \frac{3A^2}{4B^4} = -p - \Lambda \tag{3}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{A^2}{4B^4} = -p - \Lambda \tag{4}$$

Average seale factor

$$a(t) = (AB^2)^{\frac{1}{3}} \tag{5}$$

Hubble parameter

$$H = \frac{\dot{a}}{a} \tag{6}$$

Deceleration parameter

$$q = -\frac{a\ddot{a}}{\dot{a}^2} \tag{7}$$

Energy conservation equation $T^{ij}_{;j} = 0$ leads to

$$\dot{\rho} + (\rho + p)\left(2\frac{\dot{B}}{B} + \frac{\dot{A}}{A}\right) + \dot{\Lambda} = 0 \qquad (8)$$

By assuming shear tensor σ_i proportional to expansion scalar θ we have

$$A = B^n \qquad (9)$$

where n is a constant. Now introducing fractional derivatives

$$\dot{B} = (D_t^\alpha B)$$

and

$$\ddot{B} = [D_t^\alpha(D_t^\alpha B)]$$

Here we have choosen B as a function of t so let us assume

$$B = B_0 t^m$$

then

$$\frac{\dot{B}}{B} = |D_t^\alpha B_0 t^m| = \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} t^{m-\alpha} \qquad (10)$$

$$\frac{\ddot{B}}{B} = D_t^\alpha[D_t^\alpha B_0 t^m] = \frac{\Gamma(m+1)}{\Gamma(m+1-2\alpha)} t^{m-2\alpha}$$

Now equation (3) and (4)

$$\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} - \frac{3A^2}{B^4} - \frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \frac{\dot{A}\dot{B}}{AB} - \frac{A^2}{4B^4} = 0 \qquad (11)$$

from equation (9), we have

$$\frac{\dot{A}}{A} = n\frac{\dot{B}}{B}$$

and

$$\frac{\ddot{A}}{A} = n\frac{\ddot{B}}{B} - \frac{n\dot{B}^2}{B^2} + \frac{\dot{A}^2}{A^2} = \frac{n\ddot{B}}{B} - n(1-n)\frac{\dot{B}^2}{B^2} \qquad (12)$$

Using equation (12) in equation (11), we get

$$B_0 = D^{\frac{1}{2(n-2)}} t^{\frac{-(\alpha+mn-2m)}{(n-2)}} \qquad (13)$$

using B_0, B becomes

$$B = D^{\frac{1}{2(n-2)}} t^{\frac{-\alpha}{n-2}} \tag{14}$$

$$A = B^{n} = D^{\frac{n}{2(n-2)}} t^{\frac{-n\alpha}{n-2}} \tag{15}$$

where

$$D = \left(\frac{(1-n)\Gamma(m+1)}{\Gamma(m+1-2\alpha)} + \frac{(1-n^2)(\Gamma(m+1))^2}{(\Gamma(m+1-\alpha))^2} \right)$$

metric (1) reduces to

$$ds^2 = -dt^2 + D^{\frac{n}{(n-2)}} t^{\frac{-2n\alpha}{n-2}} (dy + xdz)^2 + D^{\frac{1}{n-2}} t^{\frac{-2\alpha}{n-2}} (dx^2 + dz^2) \tag{16}$$

Now average scale tensor

$$a = (AB^2)^{\frac{1}{3}} = [D^{\frac{n+2}{2(n-2)}} t^{\frac{-\alpha(n+2)}{n-2}}]^{\frac{1}{3}} \tag{17}$$

Using the above results, the Hubble parameter, Deceleration parameter, Energy density, Pressure and the Cosmological constant are given as

$$H = \frac{\dot{a}}{a} = -\frac{\alpha(n+2)}{3(n-2)} t^{-1} \tag{18}$$

$$q = -\left(\frac{3(n-2)}{\alpha(n+2)} + 1 \right) \tag{19}$$

$$\rho = \frac{1}{1+\omega} \left(\left(\frac{-2\alpha}{n-2} + \frac{2\alpha^2(n-1)}{(n-2)^2} \right) t^{-2} + \frac{D}{2} t^{-2\alpha} \right) \tag{20}$$

$$p = \frac{\omega}{1+\omega} \left(\left(\frac{-2\alpha}{n-2} + \frac{2\alpha^2(n-1)}{(n-2)^2} \right) t^{-2} + \frac{D}{2} t^{-2\alpha} \right) \tag{21}$$

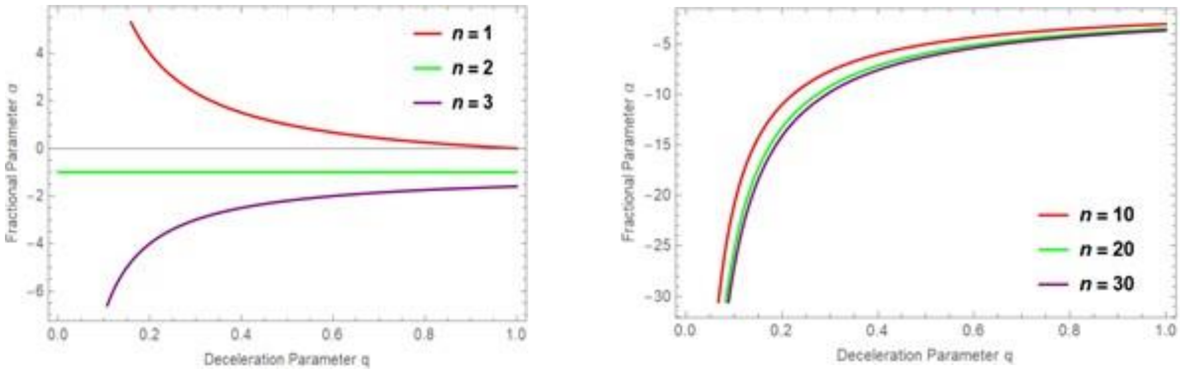
$$\Lambda = \frac{3+\omega}{4(1+\omega)} D t^{-2\alpha} + \frac{1}{(1+\omega)} \left(\frac{-2\alpha}{n-2} + \frac{2\alpha^2(n-1)}{(n-2)^2} - \frac{(2n+1)\alpha^2}{(n-2)^2} \right) t^{-2} \tag{22}$$

RESULT AND DISCUSSION

In this paper we have presented few main results of studies on Bianchi type II cosmological model with fractional derivatives and found that they satisfy the general theory of gravitation. The deceleration parameter is found as a constant but function of fractional coefficient, which shown by the table below:

α	0	.2	.4	.6	.8	1
q at n=1	∞	4	1.5	0.666	0.25	0
q at n=2	-1	1	-1	-1	-1	-1
q at n=3	$-\infty$	-4	-2.5	-2	-1.75	-1.6
q at n=10	$-\infty$	-11	-6	-4.3333	-3.5	-3

q at n=20	-∞	-13.2727	-7.1363	-5.0909	-4.0681	-3.4545
q at n=30	-∞	-14.125	-7.5625	-5.375	-4.2812	-3.625



(a) Fig 1 (b) Fig 1

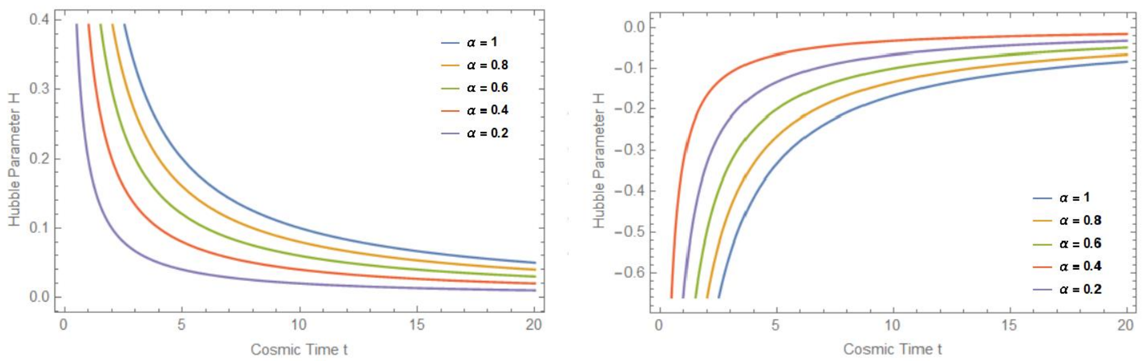
Figure 1: Graph between Deceleration parameter (q) with fractional parameter (α) at n=1, 2, 3 and 10, 20, 30.

From the figure 1, we can conclude that

For $n < 2$ ((a)Fig.1), the classical model ($\alpha = 1$) shows expanding behaviour with constant rate but as $\alpha \to 0$, the universe shows decelerating expansion.

For $n = 2$ ((a)Fig.1), universe shows constant expansion unaffected by the value of α.

For $n > 2$ ((a and b)Fig.1), universe is expanding at super exponential rate as $\alpha \to 0$.



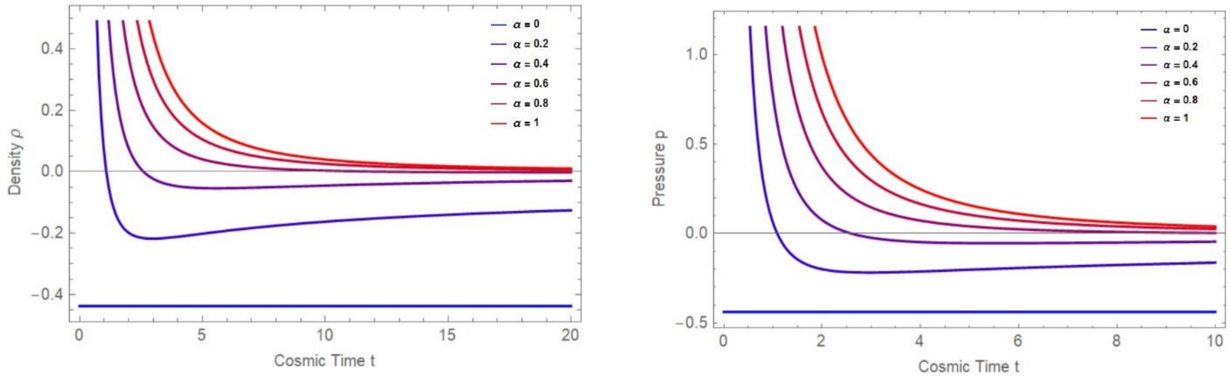
(a) Fig 2 (b) Fig 2

Figure 2: Graph between Hubble parameter (H) and cosmic time (t) at $n < 2$ and $n > 3$

Part (a) of Figure 2 shows that as $t \to 0$, then $H \to \infty$ i.e. at the time of big bang expansion rate is very large, for evolution during the time it decreases with time and converges to a constant value. Part
(b) Fig 2 shows the same behaviour that it gives the time would take rewind the universe backwards to the Big Bang assuming that the Hubble constant has been constant throughout the existence of the universe. Again both the parts of Figure 2 show that the behaviour of Hubble parameter is unaffected from fractional parameter.

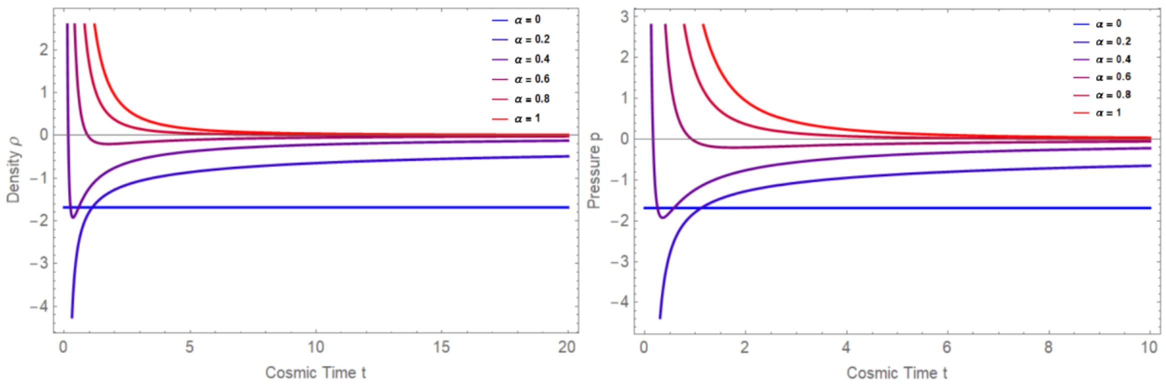
The equation of state (EOS) parameter is one of the possible sources driving the cosmic expansion and generally given by a dimensionless integer that represents the relationship between a fluid's pressure and energy density for a perfect fluid defined as $p = \omega\rho$, where $0 \leq \omega \leq 1$. Here we have discussed few cases as follows:

CASE 1 For stiff fluid dominated universe i,e when ω = 1



(a) Fig 3 Between ρ and t (b) Fig 3 Between p and t

Figure 3: Graph of energy density and fluid pressure with respect to cosmic time (t) at $n = 1.5$ and $m = 0.5$



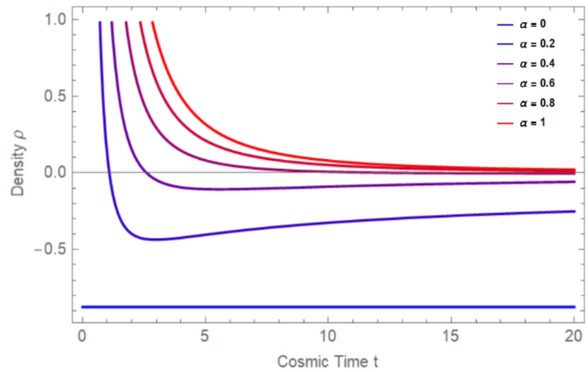
(a) Fig 4 Between ρ and t (b) Fig 4 Between p and t

Figure 4: Graph of energy density and fluid pressure with respect to cosmic time (t) at $n = 2.5$ and $m = 0.5$

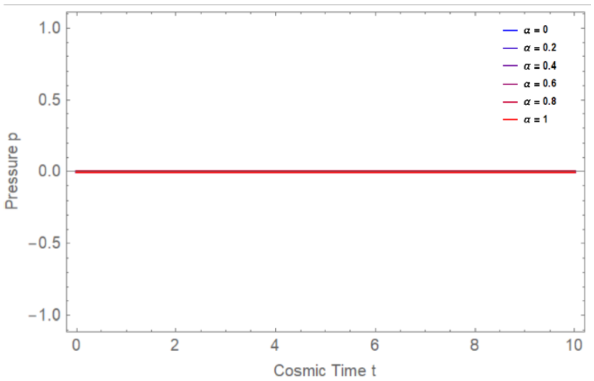
In this case we can see ((a)Fig-3) that for $n < 2$, the model exist only when the fraction parameter lies between $0.6 \leq \alpha < 1$, and for $0 < \alpha < 0.6$, model shows nonexistence. Similarly for $n > 2$ ((a)Fig-4), the range of fractional parameter reduces to $0.8 \leq \alpha < 1$, otherwise model shows nonexistence.

CASE 2 For Dust fluid dominated universe i,e when ω = 0

In the case of dust fluid the range of fraction parameter lies between $0.6 \leq \alpha < 1$ for $n < 2$ and $0.8 \leq \alpha < 1$ for $n > 2$, otherwise model shows nonexistence.

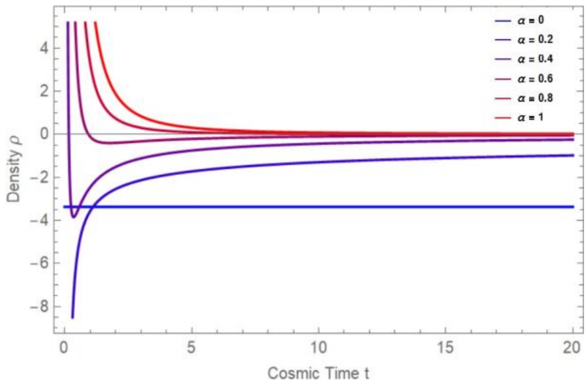


(a) Fig 5 Between ρ and t

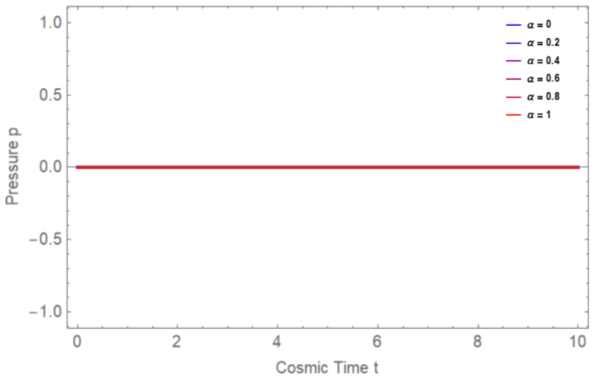


(b) Fig 5 Between p and t

Figure 5: Graph of energy density and fluid pressure with respect to cosmic time (t) at $n = 1.5$ and $m = 0.5$



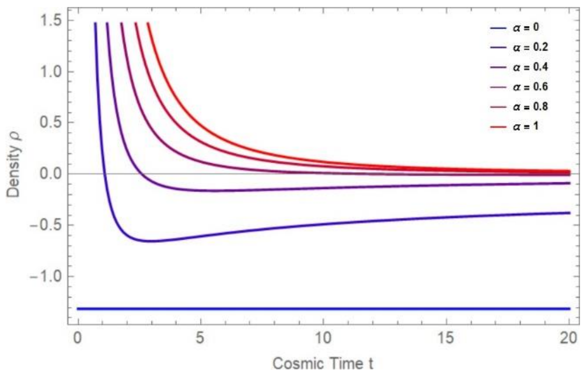
(a) Fig 6 Between ρ and t



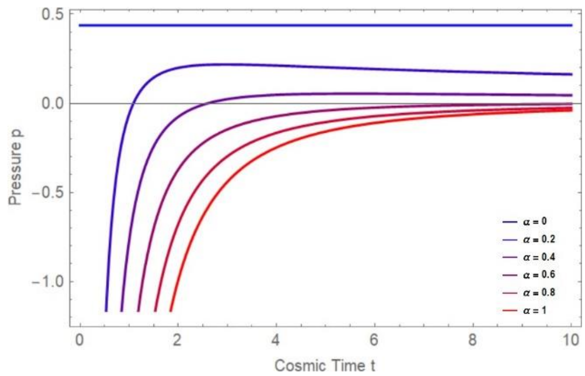
(b) Fig 6 Between p and t

Figure 6: Graph of energy density and fluid pressure with respect to cosmic time (t) at $n = 2.5$ and $m = 0.5$

CASE 3 For radiation i.e $\omega = -1$, at n = 1.5 and m = 0.5

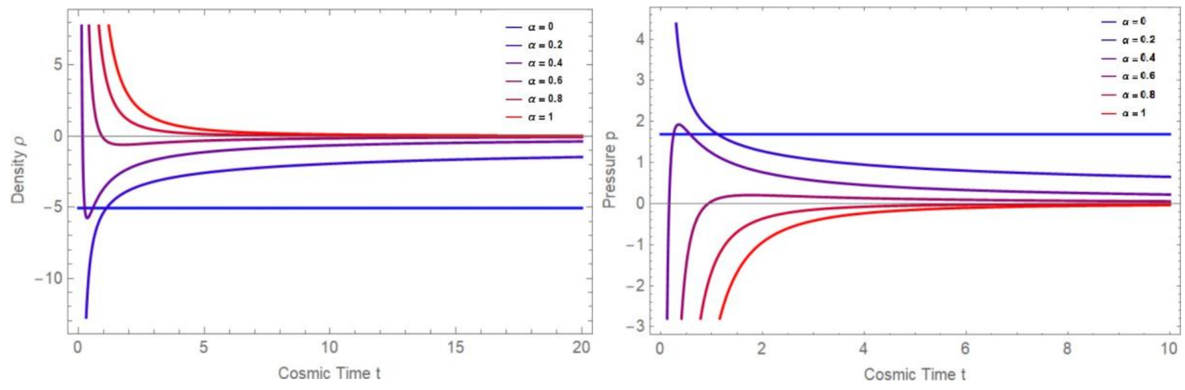


(a) Fig 7 Between ρ and t



(b) Fig 7 Between p and t

Figure 7: Graph of energy density and fluid pressure with respect to cosmic time (t) at $n = 1.5$ and $m = 0.5$



(a) Fig 8 Between ρ and t

(b) Fig 8 Between p and t

Figure 8: Graph of energy density and fluid pressure with respect to cosmic time (t) at $n = 2.5$ and $m = 0.5$

CONCLUSION

This leads us to the conclusion that the model is only valid when the fraction parameter is between $0.6 \leq \alpha < 1$, and and that the model is nonexistent for other values. It is also unaffected by the EoS parameter's value. This may play an important role in choosing the fractional parameter for the existence of the assumed model. The cosmological constant is in quadratic form, which shows that the cosmological term is decreasing function of time. Thus, the model represents shearing and nonrotating universe.

Data availability
Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Ethical Approval: The current study is not based on human or animal study, so no consent required.
Conflicts of Interest: The authors declare no conflict of interest.

Author Contributions
Conceptualization, Investigation and Methodology, R.G.; Fractional Conceptualization and Methodology, E.M. and M.M.; Writing—original draft, data analysis and Improvisations, R.G.; All authors have read and agreed to the published version of the manuscript.

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REFERENCES

1. G S Khadekar and N V Gharad: Bianchi Type-II Cosmological Model with a Linear Equation of State in Vsl Theory, *Journal of Dynamical Systems and Geometric Theories*, 11 75 (2013).
2. D R K Reddy, Y Aditya and K D Naidu: Dynamics of Bianchi type-II Anisotropic Dark Energy Cosmological Model in the Presence of Scalarmeson Fields, *Canadian Journal of Physics*, 97 9 932-937 (2019).
3. D M Gusu and M Vijaya Santhi: Analysis of Bianchi Type V Holographic Dark Energy Models in General Relativity and Lyra's Geometry, *Advances in High Energy Physics*,

2021 8818590 (2021).

4. J K Singh and N K Sharma: Anisotropic Dark Energy Bianchi Type-II Cosmological Models in Lyra Geometry, *International Journal of Theoretical Physics*, 53 1375 (2014).
5. J M Cerverõ: Time-Dependent Cosmological Constant, *General Relativity and Gravitation*, 10 503 (1979).
6. A Pradhan, D Srivastava and G S Khadekar: Can Bianchi Type-II Cosmological Models with a Decay Law for Λ Term be Compatible with Recent Observations?, *Romanian Reports in Physics*, 60 3 (2008).
7. R K Tiwari, D Tiwari and P Shukla: LRS Bianchi Type-II Cosmological Model with a Decaying Lambda Term, *Chinese Physics Letters*, XIII (2012).
8. A Banerjee, S B Duttachoudhury and A K Sanyal: Bianchi Type-II Cosmological Model with Viscous Fluid, *General Relativity and Gravitation*, 18 461 (1986).
9. L Kantha: A Time-Dependent Λ and G Cosmological Model Consistent with Cosmological Con-straints, *Advances in Astronomy*, 2016 9743970 (2016).
10. V F Panov, E V Kuvshinova, D M Yanishevsky and O V Sandakova: Bianchi type II cosmological model of the Universe's evolution, *International Journal of Geometric Methods in Modern Physics*, 15 (2018).
11. R Bali: Creation field cosmological models for dust distribution and timedependent cosmological term Λ in Bianchi type II space-time, *International Journal of Geometric Methods in Modern Physics*, 19 2250233 (2022).
12. R K Tiwari, A H A Alfedeel, D Sofuoğlu, A Abebe, E I Hassan and B K Shukla: A Cosmo-logical Model with Time Dependent Λ, G and Viscous Fluid in General Relativity, *Frontiers in Astronomy and Space Sciences*, 9 965652 (2022).
13. M D Roberts: Fractional Derivative Cosmology, *SOP Transactions on Theoretical Physics*, 2014 2 (2009).
14. El-Nabulsi Rami Ahmad: Fractional Unstable Euclidean Universe, *Electronic Journal of Theoret-ical Physics*, 2 1 (2005).
15. V K Shchigolev: Cosmological Models with Fractional Derivatives and Fractional Action, *Com-munications in Theoretical Physics*, 29 389 (2011).
16. V K Shchigolev: Cosmic Evolution in Fractional Action Cosmology, *Modern Physics Letters A*, 14 28 (2013).
17. V K Shchigolev: Testing Fractional Action Cosmology, *European Physical Journal Plus*, 131 256 (2016).
18. H Ye, J Gao and Y Ding: A Generalized Gronwall Inequality and its Application to a Fractional Differential Equation, *J. Math. Anal. Appl.*, 328 1075 (2007).
19. U Debnath, S Chattopadhyay and M Jamil: Fractional Action Cosmology: Some Dark Energy Models in Emergent, Logamediate, and Intermediate Scenarios of the Universe, *Journal of Theo-retical and Applied Physics*, 7 25 (2013).
20. U Debnath, S Chattopadhyay and M Jamil: Fractional Action Cosmology: Some Dark Energy Models in Emergent, Logamediate, and Intermediate Scenarios of the Universe, *Journal of Theo-retical and Applied Physics*, 7 25 (2013).
21. I Torres, J C Fabris, O F Piattella and A B Batistay: Quantum Cosmology of Fab Four John Theory with Conformable Fractional Derivative, *General Relativity and Quantum Cosmology*, 4 50 (2020).
22. V K Shchigolev: Fractional-Order Derivatives in Cosmological Models of Accelerated

Expansion, *Modern Physics Letters A*, 14 36 (2021).

23. E Barrientos, S Mendoza and P Padilla: Extending Friedmann Equations Using Fractional Deriva-tives Using a Last Step Modification Technique: The Case of a Matter Dominated Accelerated Expanding Universe, *Symmetry*, 13 174 (2021).
24. I Podlubny: Fractional Differential Equations-An Introduction to Fractional Derivatives, Frac-tional Differential Equations, to Methods of Their Solution and Some of Their Applications, *Fractional Differential Equations*, (Elsevier Academic Press) (1998).
25. R Gupta, E Mittal and M Menon: Bianchi Type-II Fractional Cosmological Model with Variable Lambda, *Research Square*, (https://doi.org/10.21203/rs.3.rs-2352473/v1) (2023).