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RESEARCH ARTICLE

STABILITY ANALYSIS OF THE ELLIPTIC SITNIKOV PROBLEM USING THE MONODROMY MATRIX AND FLOQUET MULTIPLIERS

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ABSTRACT

This study investigates the stability and instability of solutions of the Sitnikov problem using the monodromy matrix and Fluquet multiplier approach. The variational equations associated with the periodic solutions were analysed, and the corresponding monodromy matrix was constructed to determine the Floquet multipliers and assess the stability of the motion. Numerical simulations were carried out in Mathcad for different values of the orbital eccentricity. The results show that the periodic solution is stable in the circular case, where all Floquet multipliers lie on the unit circle. However, as the eccentricity increases, the Floquet multipliers gradually move away from the unit circle, indicating a transition from stable to unstable motion. The magnitude of the dominant eigenvalues increases with eccentricity, demonstrating that higher eccentricities lead to stronger instability of the periodic solutions. Phase portraits further illustrate the change in the qualitative behaviour of the solutions as the eccentricity varies. These findings confirm that orbital eccentricity plays a significant role in determining the stability characteristics of the Sitnikov problem and demonstrate the effectiveness of the monodromy matrix and Floquet theory as tools for stability analysis..

Keywords: Sitnikov problem, Monodromy matrix, Floquet multipliers, Stability analysis

INTRODUCTION

The Sitnikov problem describes the motion of a particle of negligible mass attracted by two equal masses $m_1 = m_2 = 1/2$. The primaries m_1 *and* m_2 move on the plane x,y, following an elliptic motion with eccentricity $e \in [0,1)$, while the massless body m_3 performs motion along an axis perpendicular to the primary orbit plane through the barycentre of the primaries. The minimal period of the elliptic motion is 2π if the gravitational constant is assumed to be $G = 1$. If z denotes the position of the particle m_3 in a coordinate system with origin at the centre of mass of the primaries, then the equation of motion in the Sitnikov problem becomes

$$\ddot{z} + \frac{z}{(z^2+r(t,e)^2)^{3/2}} = 0 \tag{1.1}$$

where z is the distance from the center of the orbit to m_3, $\ddot{z}$ is acceleration, e is eccentricity and $r(t,e)$ is the distance of the primaries to their center of mass and it is given by

$$r(t,e) = \frac{1}{2}(1 - e\cos u(t)) \tag{1.2}$$

It can either be circular or an elliptic solution with eccentricity e = 0 or 0<e <1, respectively. We have the eccentricity anomaly u(t), which is a function of time through Kepler equation

$$u - e\sin u = t \tag{1.3}$$

Without loss of generality, when $0 \le e < 1$, we take the origin of time in such a way that at $t = 0$, the primaries are at the pericenter of the ellipse. We note that system (1.1) depends on one parameter, the eccentricity $e \in [0,1)$. When the eccentricity e is zero (that is, the primaries move on the circular orbit $r(t,e) = \frac{1}{2}$ of the Kepler problem (1.3)), (1.1) becomes the equation of motion

$$\ddot{z} = -\frac{z}{\left(z^2+\frac{1}{4}\right)^{3/2}} \tag{1.4}$$

for the circular Sitnikov problem

More information can be found in [5] and [8]. Following the computational methodology proposed in [1], the stability of periodic solutions of the Sitnikov problem is investigated through the integration of the associated variational equations over one forcing period. The resulting fundamental matrix evaluated after one period defines the monodromy matrix, whose eigenvalues (Floquet multipliers) proved a complete characterization of the local stability of the periodic orbit. Although originally developed for nonlinear rotor dynamic systems, the method applies directly to the periodically forced equations of the Sitnikov problem because both systems satisfy the assumptions of Floquet theory. Recently, [3] proposed a Koopman-Hill framework that significantly advances the numerical bifurcation analysis of nonlinear dynamical systems. Their method combines the classical monodromy matrix approach with the Koopman-Hill projection, thereby providing an efficient computational framework for Floquet stability analysis. In particular, the approach enables the computation of the monodromy matrix while improving the detection and localization of codimension-1 and -2 bifurcations through a hybrid time and frequency-domain strategy. The authors demonstrated that the Koopman-Hill projection reduces computational cost and improves numerical accuracy compared with conventional Hill-based methods. However, unlike Bayer and Alcorta, whose emphasis is on developing an efficient computational framework for bifurcation detection in general nonlinear dynamical systems, this paper focuses specifically on the Sitnikov problem. But the study of [4] and [6] stressed on the stability in Sitnikov restricted problem of three bodies when the primaries are triaxial rigid bodies and oblate spheroids respectively. Comparing it with the work of [2] and [7], where one of the key points of their numerical strategy was the computation of the Jacobian of the Poincare map (i.e, the monodromy matrix), which, by serving as iteration matrix, governs both the accuracy and robustness of the whole of their strategy. While on the other hand, [6] presented a critical time step for the numerical integration based on the Runge-Kutta method of the monodromy matrix associated with a set of n first-order linear ordinary differential equations with periodic coefficients. The purpose of this work is to show the stability and instability of solutions in the Sitnikov problem at a certain value of eccentricity.

This paper is divided into sections. Section 2 is the definition and theorems which was used in the result while section 3 is the result obtained with numerical simulations and section 4 is conclusion.

Preliminary

Theorem 1. For each integer $m \geq 1$, there exists a unique solution $z(t)$ of (1.1) satisfying the conditions,

$$z(t+2mt) = z(t), z(-t) = -z(t), t \in \mathbb{R} \quad (2.1)$$

$$z(t) > 0, t \in [0, m\pi] \text{ [10]} \quad (2.2)$$

The variational equation at the center of mass $z = 0$ will play an important role; it is the equation of Hill's type

$$\ddot{\xi} + \frac{1}{r(t,e)^3}\,\xi = 0 \quad (2.3)$$

Theorem 2. Assume that $m \geq 1$ *and* $N \geq 0$ are given integers. Then the following statement are equivalent:

i. There exist a solution of (1.1) satisfying the condition in (2.1) and having exactly N zero in the interval $[0, m\pi]$.
ii. The solution $\xi(t)$ of (2.3) with initial conditions $\xi(0) = 0$, $\xi(0) = 1$ has more than N zero in $[0, m\pi]$ [10]

Lemma 1. The solution of the Sitnikov Problem (1.1) is periodic and stable if and only if the eccentricity $e = 0$.

Linearized stability

We consider the following stability criterion for Hill's equations:
Assume that $a \in C(T)$ satisfies

$$n^2 < \alpha \leq a(t) \leq \beta < \left(n + \frac{1}{2}\right)^2. \quad (2.4)$$

Then $\ddot{z} + a(t)z = 0$ is stable.

Let $\Phi(t) = \begin{pmatrix} z_1(t) & z_2(t) \\ \dot{z}_1(t) & \dot{z}_2(t) \end{pmatrix}$

be the matrix solution with solutions $z_1(t)$, $z_2(t)$ satisfying

$z_1(0) = \dot{z}_2(0) = 1, \dot{z}_1(0) = z_2(0) = 0.$

The stability of the equation is equivalent to the stability of the monodromy matrix

$M = \Phi(2\pi) \in Sp(\mathbb{R}^2)$. We shall prove that the trace satisfies $|trM| < 2$ and so M is elliptic.

Theorem 3. There exists a number $\theta^*, {}^{\pi}/_{2} < \theta^* < {}^{2\pi}/_{3}$, such that if (2.3) had Floquet

multipliers $e^{\pm i\theta}$ with

then $z = 0$ is of twist type. [7].

We use the Floquet multipliers to achieve the following in the Sitnikov problem:

i. That the Floquet multipliers $|\mu_i| \leq 1$ confirms stability.
ii. That the Floquet multipliers $|\mu_i| > 1$ confirms instability.
iii. That the Floquet multipliers $|\mu_i| = 1$ confirms neutral direction.

RESULT

Linearized stability of the Sitnikov problem with Monodromy matrix at $\boldsymbol{e} = \mathbf{0}$

From equation 2.3,

$$\text{Let } a(t) = \frac{1}{r(t,e)^3} \tag{3.1}$$

Such that by linearization,

$$\dot{x} = -a(t)z \tag{3.2}$$

Let $A(t) = -a(t)$ such that;

$$A(t) = \begin{pmatrix} 0 & 1 \\ -a(t) & 0 \end{pmatrix} \tag{3.3}$$

We derive the fundamental matrix of the linear system by first finding the eigenvalues. The characteristic equation becomes

$$\det(\mathrm{A} - \lambda\mathrm{I}) = \begin{vmatrix} -\lambda & 1 \\ -8 & -\lambda \end{vmatrix}$$

$$= \lambda^2 + 8 = 0 \tag{3.4}$$

Hence,

$$\lambda = \pm 2\sqrt{2}\mathrm{i}. \tag{3.5}$$

Computing the matrix exponential, we have that

$$\mathrm{A}^2 = \begin{pmatrix} 0 & 1 \\ -8 & 0 \end{pmatrix}^2$$

$$= \begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix}$$

$$= -8\mathrm{I}. \tag{3.6}$$

Let

$$\mathrm{w} = \sqrt{8}$$

$$= 2\sqrt{2} \tag{3.7}$$

Since $\mathrm{A}^2 = -\mathrm{w}^2\mathrm{I}$,

$$e^{At} = I\cos(wt) + \frac{A}{w}\sin(wt)$$

Substituting A and $\mathrm{w} = 2\sqrt{2}$, we have

$$\phi(t) = \begin{pmatrix} \cos\left(2\sqrt{2}\,t\right) & \frac{1}{2\sqrt{2}}\sin\left(2\sqrt{2}\,t\right) \\ -2\sqrt{2}\sin\left(2\sqrt{2}\,t\right) & \cos\left(2\sqrt{2}\,t\right) \end{pmatrix} \tag{3.8}$$

The fundamental matrix therefore, satisfies:

$$\phi(t) = I.$$

Finding the Monodromy matrix and the Floquet multipliers of the above system, given the eigenvalues

$$\lambda = \pm 2\sqrt{2i},$$

the angular frequency becomes $w = 2\sqrt{2}$, such that the period T becomes;

$$T = \frac{2\pi}{w} = \frac{2\pi}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}$$

Thus, every solution repeats after

$$T = \frac{\pi}{\sqrt{2}}. \tag{3.10}$$

We evaluate the fundamental matrix at one period by substituting

$$t = T = \frac{\pi}{\sqrt{2}}.$$

Then

$$2\sqrt{2}T = 2\sqrt{2}\left(\frac{\pi}{\sqrt{2}}\right) = 2\pi.$$

Since $\cos(2\pi) = 1, \sin(2\pi) = 0$, we obtain

$$\phi(T) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.11}$$

Hence, the monodromy matrix becomes

$$M = \phi(T) = I. \tag{3.12}$$

The Floquet multipliers are the eigenvalues of the Monodromy matrix.

Since;

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

Its characteristic polynomial is

$$\det(M - \mu I) = (1 - \mu)^2 = 0.$$

Therefore,

$$\mu_1 = \mu_2 = 1.$$

Stability analysis

Based on Floquet stability criterion, the periodic solution is Lyapunov stable but not asymptotically stable.

The trajectories neither decay toward the equilibrium nor diverge away from it; instead, they remain on closed periodic orbits around the origin.

Linearized stability or instability of the Sitnikov problem with Monodromy matrix at $e > 0$

$$\text{At e} = 0.1,\ \mathrm{r}(t,e) = \frac{9}{20},\ \mathrm{a}(t,e) = \frac{8000}{729}.$$

Since every entry of A is constant, the matrix satisfies

$A(t+T) = A(t)$.

We previously found

$$\phi(\mathrm{t}) = \begin{pmatrix} \cos(w\,t) & \frac{1}{w}\sin(w\,t) \\ -w\sin(w\,t) & \cos(w\,t) \end{pmatrix} \tag{3.13}$$

such that

$$\phi(\mathrm{t}) = \begin{pmatrix} \cos\left(\frac{40\sqrt{5}}{27}\,t\right) & \frac{27}{40\sqrt{5}}\sin\left(\frac{40\sqrt{5}}{27}\,t\right) \\ -\frac{40\sqrt{5}}{27}\sin\left(\frac{40\sqrt{5}}{27}\,t\right) & \cos\left(\frac{40\sqrt{5}}{27}\,t\right) \end{pmatrix}$$

where,

$$w = \frac{40\sqrt{5}}{27}$$

The sine and cosine functions have period $T = \frac{2\pi}{w}$.

Substituting $w = \frac{40\sqrt{5}}{27}$, we have

$$T = \frac{2\pi}{40\sqrt{5}/27} = \frac{27\pi}{20\sqrt{5}} \tag{3.14}$$

The monodromy matrix is

$$M = \phi(\mathrm{T}).$$

Since

$$wT = 2\pi, \tag{3.15}$$

We have

$\cos(2\pi) = 1,\ \sin(2\pi) = 0.$

Hence,

$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Therefore

$M = I.$

The same process occurs because the system is conservative (a simple harmonic oscillator), the solutions are

$$z(t) = C_1 \cos(wt) + C_2 \sin(wt) \tag{3.16}$$

Which remain bounded for all time.

When the amplitude is not constant, then the behaviour of the system changes significantly.

That is

$$z(t) = Ae^{\alpha t} \cos(wt)\,, \alpha > 0. \tag{3.17}$$

$|\mu_i| > 1.$

Suppose the monodromy matrix has Floquet multipliers

$$\mu_1 = 1.35, \mu_2 = \frac{1}{1.35}$$

Since

$$|\mu_1| = 1.35 > 1, \tag{3.18}$$

The oscillation amplitude grows after each period.

Let the initial amplitude be A_0, then after n periods it is approximately

$$A_n \approx A_0(1.35)^n,$$

So the orbit moves farther away from the periodic solution. This then reveal instability.

Suppose the amplitude slowly increases, we then suppose the oscillation is

$$z(t) = e^{\alpha t} \cos(wt), \text{ where } \alpha > 0.$$

The corresponding matrix becomes

$$A = \begin{pmatrix} 0 & 1 \\ -w^2 & 2\alpha \end{pmatrix} \tag{3.19}$$

The eigenvalues are

$$\lambda = \alpha \pm \mathrm{i}w. \tag{3.20}$$

Now the fundamental matrix becomes

$$\phi(t) = e^{\alpha t} \begin{pmatrix} \cos(w\,t) & \frac{1}{w}\sin(w\,t) \\ -w\sin(w\,t) & \cos(w\,t) \end{pmatrix} \tag{3.21}$$

we notice the factor $e^{\alpha t}$.

This exactly causes the amplitude to grow.

After one period,

$$\mathrm{T} = \frac{2\pi}{w},$$

The monodromy matrix becomes

$$M = e^{\alpha T} I. \tag{3.22}$$

Its eigenvalues are

$$\mu_1 = \mu_2 = e^{\alpha T}.$$

Since

$$e^{\alpha T} > 1,$$

The Floquet multipliers satisfy

$$|\mu| > 1.$$

Hence; the periodic solution is unstable.

Suppose

$$\alpha = 0.10, \text{ then } w = \frac{40\sqrt{5}}{27} \approx 3.312,$$

Hence;

$$T = \frac{2\pi}{3.312} \approx 1.897.$$

Therefore,

$$\mu = e^{0.10(1.897)} = e^{0.1897} \approx 1.209.$$

This reveal that every period the amplitude increases by about 20.9%.

Let us assume that the initial amplitude is $A_0 = 1$, then;

Table 1

Period	Amplitude
0	1.00
1	1.209
2	1.462
3	1.768
4	2.138
5	2.584

The oscillation grows larger after each cycle, demonstrating instability.

SIMULATION AND ANALYSIS OF THE SITNIKOV PROBLEM [11]

The Sitnikov problem is: $\ddot{z} + \frac{z}{(z^2+r(t,e)^2)^{3/2}} = 0 \qquad z(0) = 0, \dot{z}(0) = 1.$

with $r(t,e) = 0.5\big(1 - e\cos\big(u(t)\big)\big).$

Over the time interval $[0, 50]$, we produce the plots with grid lines of:

1. The trajectory $z(t)$
2. The velocity $\dot{z}(t)$
3. The phase portrait. That is $\dot{z}$ against z.

We display the plots with grid lines for;

- $u(t) = 0$, $e = 0.5$ –Constant forcing with moderate eccentricity
- $u(t) = t$, $e = 0.5$ –Linear increasing forcing
- $u(t) = \sqrt{t}$, $e = 0.5$ –Square root forcing (slower increase)
- $u(t) = 0$, $e = 0.9$ –Constant forcing with high eccentricity
- $u(t) = \sqrt{t}$, $e = 0.9$ –Square root forcing with high eccentricity

RESULTS

Figure 1

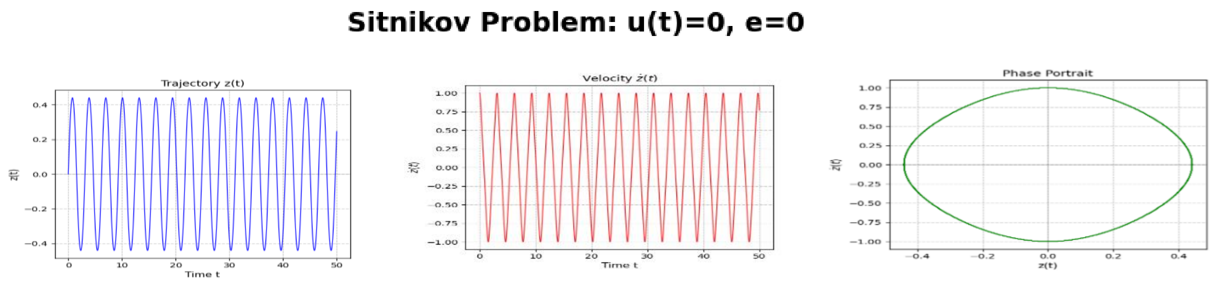


Figure 2

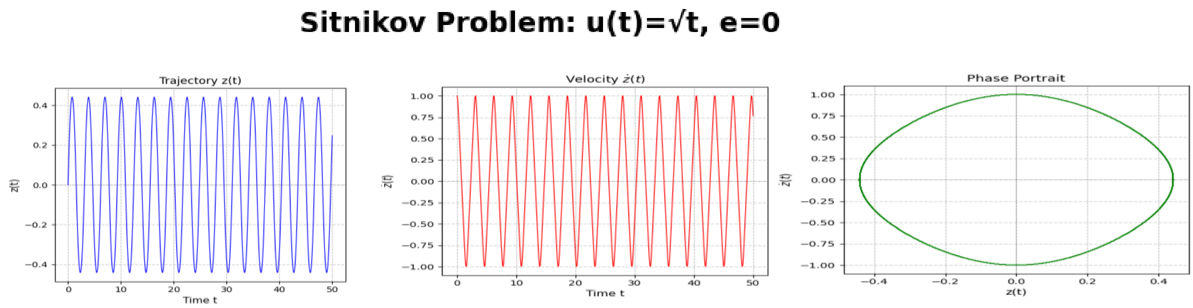


Figure 3

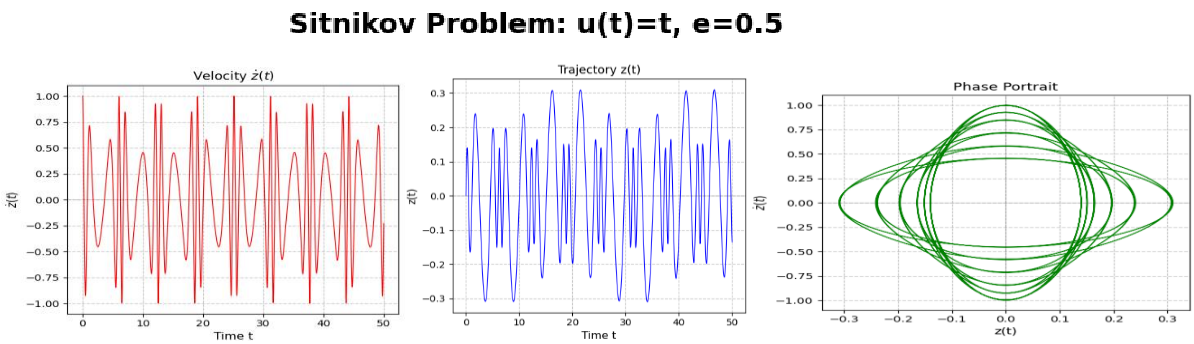


Figure 4

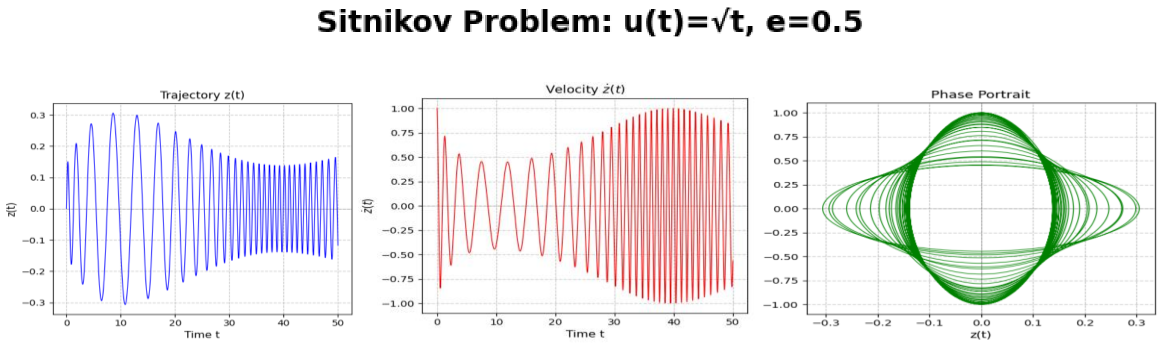


Figure 5

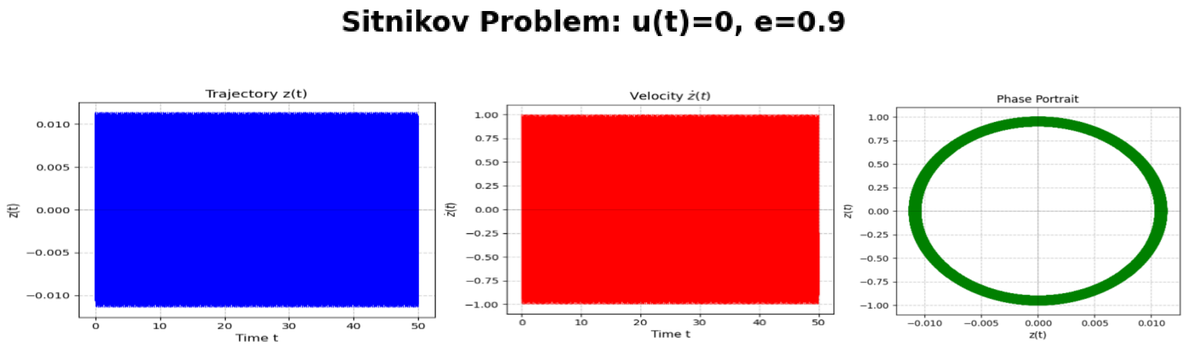
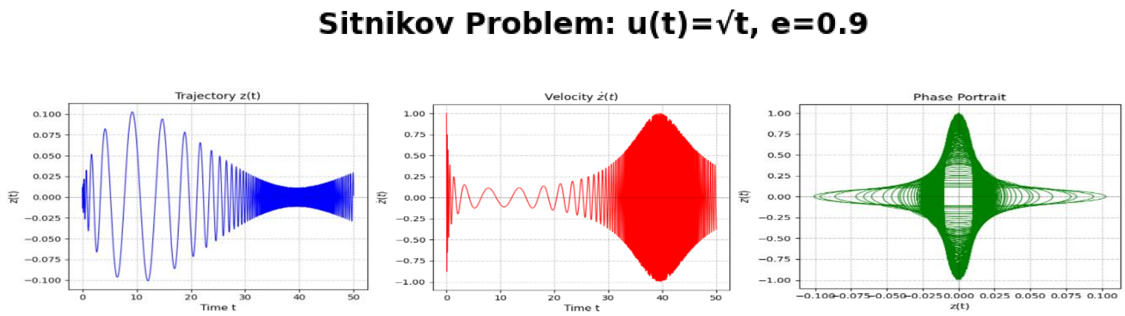


Figure 6



Brief observations

For $u(t) = 0$, the motion is strongly oscillatory around the origin and stable showing conservation of the system.

For $u(t) = t$ and $u(t) = \sqrt{t}$, the forcing through $r(t, e)$ modifies the amplitude and frequency of oscillation resulting to instability of the system.

The phase portraits show bounded oscillatory trajectories with changing geometry depending on the choice of $u(t)$.

CONCLUSION

This study investigated the stability of periodic solutions of the Sitnikov problem using the monodromy matrix and Floquet multiplier method. The variational equations were analysed, and numerical simulations were performed in Mathcad for different values of the orbital eccentricity. The results show that the periodic solution is stable in the circular case (e=0), where the Floquet multipliers remain on the unit circle, satisfying the stability criterion. As the eccentricity increases, the Floquet multipliers move outside the unit circle, indicating a loss of stability. Furthermore, the dominant eigenvalues increase in magnitude with increasing eccentricity, demonstrating that the degree of instability becomes more pronounced as the orbital eccentricity grows. The phase portraits also reveal significant changes in the qualitative behaviour of the trajectories, illustrating the transition from bounded stable oscillations to unstable motion as the system parameters vary. Overall, the study establishes that orbital eccentricity is a key parameter governing the stability of the Sitnikov problem. The monodromy matrix and Floquet multiplier techniques provide an effective framework for characterizing the stability and instability of periodic solutions and can be extended to investigate more complex dynamical behaviours in celestial mechanics.

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